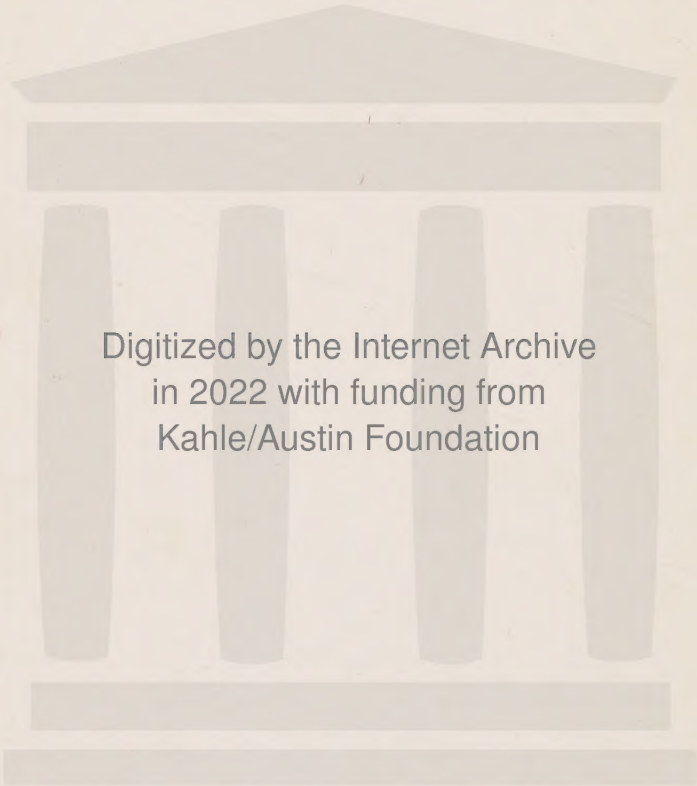

Digitized by the Internet Archive
in 2022 with funding from
Kahle/Austin Foundation

https://archive.org/details/vectoranalysis0014hbph

Moosman Kenneth
Math Dept.

KENNETH MOOSMAN
315 EAST 650 NORTH
BOUNTIFUL, UTAH

VECTOR ANALYSIS

BOOKS BY

H. B. PHILLIPS, Ph.D.

ANALYTIC GEOMETRY AND CALCULUS. Second Edition.
504 pages. 5⅝ by 8⅝. Illustrated.
Cloth.

DIFFERENTIAL EQUATIONS
125 pages. 5¼ by 7½. Illustrated.
Cloth.

VECTOR ANALYSIS
236 pages. 5¼ by 7½. Illustrated.
Cloth.

CALCULUS
353 pages. 5¼ by 7½. Illustrated.
Cloth.

ANALYTIC GEOMETRY
197 pages. 5¼ by 7½. Illustrated.
Cloth.

Published by

JOHN WILEY & SONS, Inc.

VECTOR ANALYSIS

BY

H. B. PHILLIPS, PH.D.

Professor of Mathematics
Massachusetts Institute of Technology

Fourteenth Printing

NEW YORK
JOHN WILEY & SONS, INC.
LONDON: CHAPMAN & HALL, LIMITED

COPYRIGHT, 1933
BY HENRY B. PHILLIPS

All Rights Reserved

This book or any part thereof must not be reproduced in any form without the written permission of the publisher.

FOURTEENTH PRINTING, OCTOBER, 1953

Printed in U. S. A.

PREFACE

It is the object of this book to present Vector Analysis in the form that is required for work in theoretical electricity and hydrodynamics. For these subjects what is needed is the analysis of vector fields and the study of the quantities which characterize each type of field. These quantities have essentially the same form and properties in whatever field they occur. The discussion of such matters in a separate course rather than their gradual introduction in connection with one of the typical fields has the advantage of showing which results follow merely from mathematics and which are dependent on physical hypotheses.

In broad outline the book consists of two parts. The first five chapters cover the fundamental operations and the more general properties of scalar and vector fields. The remaining chapters contain the detailed analysis of fields, the properties of potentials, and linear vector functions. In an elementary course the work might be restricted mainly to the first five chapters together with selected topics from the others.

The author is indebted to Charlotte T. Perry for assistance in the entire preparation of the manuscript and revision of the proof.

H. B. PHILLIPS.

CAMBRIDGE, MASS.
February, 1933.

CONTENTS

CHAPTER I

ELEMENTARY OPERATIONS

SECTION		PAGE
1.	Definitions	1
2.	Addition and Subtraction of Vectors	2
3.	Multiplication by a Scalar	3
4.	Linear Functions	4
5.	Rectangular Coördinates	6
6.	The Scalar Product of Two Vectors	7
7.	Work	9
8.	Vector Product	10
9.	Rotation About a Fixed Axis	12
10.	Moment of a Force	13
11.	Products Involving Three or More Vectors	15
12.	Functions of a Single Scalar Variable. Derivative	18
13.	Linear Vector Differential Equations	20
14.	Dynamics of a Particle or System of Particles	21

CHAPTER II

PARTIAL DIFFERENTIATION

15.	Functions of Position	30
16.	Continuity	31
17.	Gradient of a Scalar Function	32
18.	The Operator ∇	35
19.	Differentiation Formulas	38
20.	Functional Relations	40
21.	Divergence and Curl of a Vector	43

CHAPTER III

INTEGRATION

22.	Space Curves	49
23.	Line Integral	50
24.	Case of Line Integral Independent of Path	51
25.	Work and Potential	53
26.	Circulation	55
27.	Integration in a Plane	57

SECTION PAGE
28. Surface Integrals 59
29. Stokes's Theorem 62
30. Vortices 65
31. Flow Across a Surface 66
32. The Divergence Theorem 67
33. Sources and Sinks 70
34. Other Integration Formulas 72

CHAPTER IV

GENERAL COÖRDINATES

35. Curvilinear Coördinates 80
36. Reciprocal Systems 82
37. Orthogonal Coördinates 84
38. Cylindrical Coördinates 85
39. Spherical Coördinates 86
40. Gradient, Divergence, and Curl 87
41. Volume Integrals 90
42. Coördinates on a Surface 91
43. Surface Integrals 94

CHAPTER V

IRROTATIONAL AND SOLENOIDAL VECTORS

44. Reducible Curves and Surfaces 100
45. Irrotational Vectors 100
46. Velocity Potential 101
47. Condition of Integrability 102
48. Solenoidal Vectors 103
49. Vector Potential 104
50. Surface Discontinuities 106

CHAPTER VI

ELECTROSTATIC FIELDS

51. Introduction 111
52. Solid Angle 112
53. Point Charges 118
54. Space Charges 120
55. Improper Integrals 122
56. Values Inside the Charged Region 126
57. Higher Derivatives of Potential 130

SECTION PAGE
58. Laplace's and Poisson's Equations 131
59. Hölder Condition 132
60. Surface Charges 135
61. Region Near a Point on a Surface 135
62. Potential at Points on the Charged Surface 137
63. Electric Intensity at Points Near the Surface 138
64. Double Layer 140
65. Two-Dimensional Fields 144

CHAPTER VII

HARMONIC FUNCTIONS

66. Introduction 149
67. Laplace's Equation in Orthogonal Coördinates 150
68. Boundary Value Relations 150
69. Analytic Properties 153
70. Average Value Theorem 156
71. Infinite Regions 157
72. Maximum and Minimum Properties 160
73. Boundary Value Properties 161
74. Green's Function 162
75. The Dirichlet Problem 165
76. Poisson's Integral 168
77. The Neumann Problem 170
78. Converse of the Average Value Theorem 174
79. Representation of Harmonic Functions as Potentials 176
80. Characteristics which Determine a Vector 177

CHAPTER VIII

SCALAR AND VECTOR POTENTIALS

81. Introduction 182
82. Expression of a Scalar Function as Potential 182
83. Expression of a Vector as the Sum of an Irrotational and a Solenoidal Vector 186
84. Infinite Regions 187
85. Potentials of a Vector 188
86. Expression of the Potentials in Terms of Characteristics of the Vector 190
87. Equation of Continuity 192
88. Boundary Integrals 194
89. Irrotational Vectors. Solenoidal Vectors 195

CHAPTER IX

RETARDED POTENTIALS

SECTION PAGE

90. Variable Fields 200
91. Expression of a Scalar Function as Retarded Potential 201
92. Derivatives of Retarded Potentials 203
93. The Wave Equation 206
94. Expression of a Vector in Terms of Retarded Potentials 209
95. Expression of the Potentials in Terms of Characteristics of the Vector 210
96. Equation of Continuity 212
97. The Electromagnetic Field 213

CHAPTER X

LINEAR VECTOR FUNCTIONS

98. Linear Vector Functions 215
99. Dyadics 216
100. Conjugate Dyadics 217
101. Scalar and Vector of a Dyadic 219
102. The Idemfactor 220
103. Product of Dyadics 221
104. Derivatives of a Vector 223
105. Differentiation and Integration of Dyadics 224
106. Double Multiplication 226
107. Stress 228

INDEX 233

VECTOR ANALYSIS

CHAPTER I

ELEMENTARY OPERATIONS

1. Definitions. The quantities of physics can be divided into two classes, namely, those having magnitude only and those having magnitude and direction.

A quantity characterized by magnitude only, or magnitude and an algebraic sign, is called a **scalar.** Thus mass, time, density, coördinates are scalars. When units of measurement have been chosen, a scalar is represented by a real number and so is subject to all the laws of ordinary algebra.

A quantity which has direction as well as magnitude is called a **vector.** Force, velocity, acceleration are illustrations. Such a quantity is represented graphically by an arrow of length equal to its magnitude, pointing in the assigned direction.

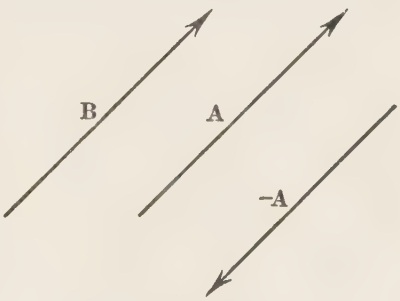


FIG. 1

Two vectors **A**, **B** are called equal,

$$\mathbf{A} = \mathbf{B},$$

if they have the same length and direction. This does not mean that the two quantities represented by **A** and **B** are in every sense equivalent. Thus, if **A** and **B** represent equal forces applied at different points of a rigid body, they may not have identical mechanical effects. But in each application quantities represented by equal vectors are in some sense equivalent, and that sense must be kept in mind when interpreting results.

The symbol $-\mathbf{A}$ is used to represent a vector having the same length as **A** but the opposite direction.

We shall find that vectors and scalars satisfy different algebraic laws. It is thus desirable to have a notation which indicates clearly which quantities are scalars and which are vectors. In most books vectors are indicated by **bold-faced** type. In this book we shall follow this convention but, since the bold-faced type is impossible to indicate in writing, we shall (with a few exceptions) also use capitals exclusively for vectors and small letters for scalars. Sometimes an arrow is placed over a symbol to indicate that it represents a vector. In particular the notation

$$\overrightarrow{PQ}$$

is used to represent the vector from P to Q.

Associated with a vector **A** is a positive scalar equal to its length, or magnitude. We shall represent this by the notation $|\mathbf{A}|$. If then a is the length of the vector **A**,

$$a = |\mathbf{A}|.$$

2. Addition and Subtraction of Vectors. Given any two vectors **A** and **B**, we can construct a vector equal to **B** with its origin coincident with the end of **A** as shown in Fig. 2. The vector from the initial point of **A** to the terminal point of this

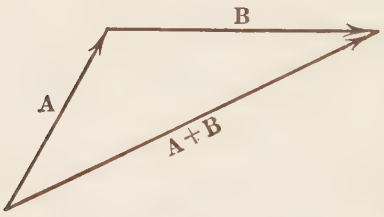


Fig. 2

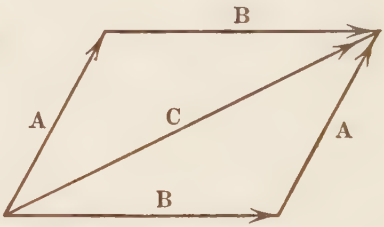


Fig. 3

vector **B** is called the sum of **A** and **B**, indicated by the notation $\mathbf{A} + \mathbf{B}$. Since the opposite sides of a parallelogram are equal and parallel, it is clear from Fig. 3 that

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \tag{1}$$

both sums being equal to the vector diagonal **C**. The sum is thus independent of the order in which the vectors are added. This is called the **commutative law** of addition.

Similarly the sum of three or more vectors is obtained by constructing a polygon (Fig. 4) having those vectors as consecutive sides and drawing a vector from the initial point of the first to the terminal point of the last. By constructing the vector diagonals $\mathbf{A} + \mathbf{B}$ and $\mathbf{B} + \mathbf{C}$ it is also clear from this diagram that

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}). \tag{2}$$

The sum is thus independent of the way its elements are associated into groups. This is called the **associative law** of addition.

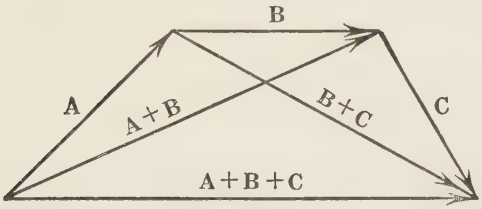


FIG. 4

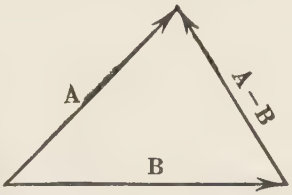


FIG. 5

If two vectors **A** and **B** are drawn from the same origin (Fig. 5) the difference $\mathbf{A} - \mathbf{B}$ is defined as the vector extending from the end of **B** to the end of **A**. This diagram also shows that

$$\mathbf{B} + (\mathbf{A} - \mathbf{B}) = \mathbf{A},$$

which indicates that parentheses can be removed and the terms cancelled as in ordinary algebra.

3. Multiplication by a Scalar. If m is a scalar and **A** is a vector, the symbol

$$m\mathbf{A} = \mathbf{A}m$$

is used to represent a vector m times as long as **A** and having the same direction if m is positive, the opposite direction if m is negative. If m, n are scalars it follows that

$$(m + n)\mathbf{A} = m\mathbf{A} + n\mathbf{A} \tag{3}$$

since both sides represent a vector $m + n$ times as long as **A** having the same direction as **A** if $m + n$ is positive, the opposite direction if $m + n$ is negative.

Similarly, if m is a scalar and **A**, **B** are vectors,

$$m(\mathbf{A} + \mathbf{B}) = m\mathbf{A} + m\mathbf{B}. \tag{4}$$

For this equation merely expresses that if the vector sides **A**, **B** and the diagonal **A** + **B** of a parallelogram (Fig. 6) are all increased in the same proportion the new vectors $m\mathbf{A}$, $m\mathbf{B}$, $m(\mathbf{A} + \mathbf{B})$ form the sides and diagonal of a similar parallelogram.

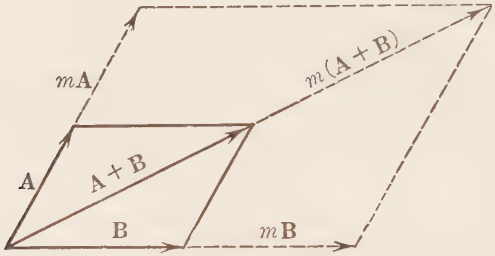


Fig. 6

Equations (3) and (4) express the **distributive law** for the multiplication of a scalar and a vector. This is the principal law of multiplication.

4. Linear Functions. An expression such as

$$m\mathbf{A} + n\mathbf{B} + p\mathbf{C},$$

obtained from a set of vectors by addition and scalar multiplication, is called a **linear function** of those vectors.

If **A** and **B** are vectors in a plane,

$$m\mathbf{A} + n\mathbf{B}$$

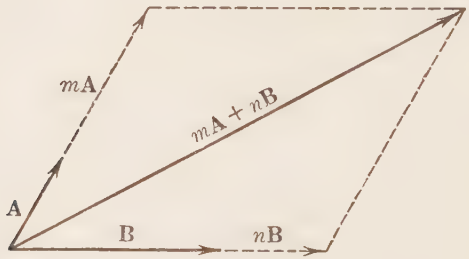


Fig. 7

is a vector in the same plane since it represents the diagonal of a parallelogram (Fig. 7) with sides parallel to **A** and **B**. Conversely, if **A** and **B** are not parallel, any vector **C** in their plane

is the diagonal of a parallelogram with sides extending along **A** and **B** and so is expressible in the form

$$\mathbf{C} = m\mathbf{A} + n\mathbf{B}.$$

Similarly, if **A**, **B**, **C** are not parallel to a plane, any vector **V** is the diagonal of a parallelepiped with edges parallel to **A**, **B**, **C**. Hence numbers m, n, p exist such that

$$\mathbf{V} = m\mathbf{A} + n\mathbf{B} + p\mathbf{C}.$$

Let **A**, **B**, **P** be the vectors from a fixed point O to points A, B, P. The vector from A to P is $\mathbf{P} - \mathbf{A}$ and that from A

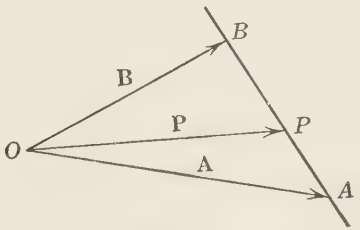


FIG. 8

to B is $\mathbf{B} - \mathbf{A}$. If P lies on the line AB and divides it in the ratio

$$\frac{AP}{AB} = m,$$

these vectors have the same ratio, that is

$$\mathbf{P} - \mathbf{A} = m(\mathbf{B} - \mathbf{A}),$$

or

$$\mathbf{P} = m\mathbf{B} + (1 - m)\mathbf{A}.$$

The vector **P** is thus a linear function of **A** and **B** with coefficients whose sum is

$$m + 1 - m = 1.$$

Conversely, if

$$\mathbf{P} = m\mathbf{A} + n\mathbf{B}$$

and $m + n = 1$, the point P lies on the line AB. For this equation is then equivalent to

$$\mathbf{P} - \mathbf{A} = n(\mathbf{B} - \mathbf{A})$$

which shows that the vector $\overrightarrow{AP}$ extends along the line AB.

Similarly, if **A**, **B**, **C**, **P** are vectors extending from a common origin O to the points A, B, C, P and

$$\mathbf{P} = m\mathbf{A} + n\mathbf{B} + p\mathbf{C},$$

the condition that P be a point in the plane ABC is

$$m + n + p = 1.$$

5. Rectangular Coördinates. A rotation about an axis is called positive, or **right-handed**, if it causes a right-handed screw to advance in the positive direction along the axis.

Let $OXYZ$ be a right-handed system of coördinate axes. By this we mean that a right-handed rotation of 90° about OZ carries OX into OY. The vectors of unit length and positive direction along OX, OY, OZ are called **i**, **j**, **k** respectively. It should be noted that the use of small letters to represent these unit vectors is an exception to the general rule that vectors are denoted by capitals.

Fig. 9

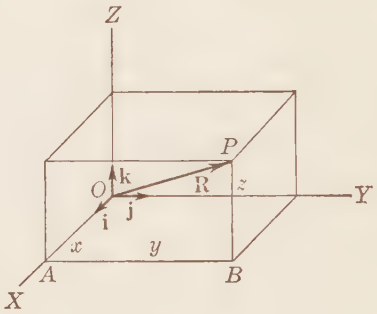


Fig. 10

Let P be the point with coördinates x, y, z. The vector $\mathbf{R} = \overrightarrow{OP}$ (Fig. 10) is the sum of three vectors $\overrightarrow{OA}$, $\overrightarrow{AB}$, $\overrightarrow{BP}$ equal to $x\mathbf{i}$, $y\mathbf{j}$, $z\mathbf{k}$. Hence

$$\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \tag{5}$$

This vector is often used to represent the point P of coördinates x, y, z.

By the **component** of a vector along a coördinate axis we mean its projection on that axis with a sign positive or negative according to the direction it extends along the axis. Thus the

vector **R** has components x, y, z. Similarly

$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

are vectors with components a_1, a_2, a_3 and b_1, b_2, b_3 respectively. The sum of these vectors is

$$\mathbf{A} + \mathbf{B} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}.$$

The components of $\mathbf{A} + \mathbf{B}$ are thus obtained by adding corresponding components of **A** and **B**.

If two vectors are equal their components are equal and conversely. This fact is often used to prove relations between vectors.

6. The Scalar Product of Two Vectors. The **scalar, or dot, product** of two vectors **A** and **B** (indicated by placing a dot between the factors) is defined as the scalar

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta, \qquad (6)$$

where θ is the angle between the two vectors.

The quantity

$$b = |\mathbf{B}| \cos\theta$$

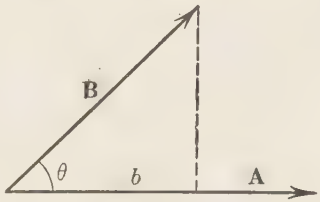


FIG. 11

is called the **component** of **B** along **A**. It is numerically equal to the projection of **B** on **A**, is positive when the projection extends in the same direction as **A**, and negative when it extends in the opposite direction. The scalar product

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta = |\mathbf{A}|\, b$$

is equal to the product of $|\mathbf{A}|$ *and the component of* **B** *along* **A**. It is also equal to the product of $|\mathbf{B}|$ and the component of **A** along **B**.

From the definition it is evident that

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}. \qquad (7)$$

That is, the scalar product is commutative.

Let b, c be the components of **B**, **C** along **A**. From Fig. 12 it is clear that $b + c$ is the component of $\mathbf{B} + \mathbf{C}$ along **A**. The equation

$$|\mathbf{A}|\,(b + c) = |\mathbf{A}|\, b + |\mathbf{A}|\, c$$

is then equivalent to

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}. \tag{8}$$

Since the products are commutative this requires also

$$(\mathbf{B} + \mathbf{C}) \cdot \mathbf{A} = \mathbf{B} \cdot \mathbf{A} + \mathbf{C} \cdot \mathbf{A}. \tag{9}$$

Equations (8) and (9) express the distributive law for scalar multiplication of vectors.

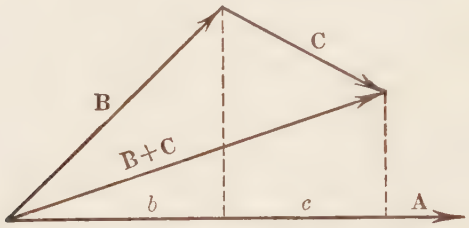


FIG. 12

If **A** and **B** are perpendicular, cos θ is zero, and

$$\mathbf{A} \cdot \mathbf{B} = 0.$$

Conversely, *if the scalar product is zero, one of the vectors is zero or else the two are perpendicular.*

Since the product may be zero when neither factor is zero, it follows that *division by a vector cannot be performed.* Thus from

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$$

we cannot conclude that

$$\mathbf{B} = \mathbf{C}$$

but merely that

$$\mathbf{A} \cdot (\mathbf{B} - \mathbf{C}) = 0$$

and so $\mathbf{B} - \mathbf{C}$ is zero or perpendicular to **A**.

If **A** and **B** have the same direction, cos $\theta = 1$ and $\mathbf{A} \cdot \mathbf{B}$ is equal to the product of the lengths of the two vectors. In particular

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2$$

is the square of the length of **A**. The product of a vector with

itself is often written as the square of the vector. Thus

$$\mathbf{A} \cdot \mathbf{A} = \mathbf{A}^2$$

but no other powers of a vector are defined.

Applying the definition of scalar product to the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ we obtain

$$\begin{aligned} \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1, \\ \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0. \end{aligned} \tag{10}$$

If

$$\begin{aligned} \mathbf{A} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \\ \mathbf{B} &= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \end{aligned}$$

by expanding and using (10) we find

$$\mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3. \tag{11}$$

7. Work. The simplest illustration of the scalar product is furnished by mechanical work. When the point of application of a force $\mathbf{F}$ experiences a displacement represented in magnitude

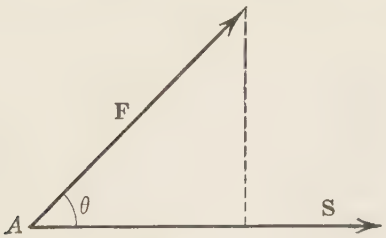


FIG. 13

and direction by the vector $\mathbf{S}$ (Fig. 13), the work done by the force is defined as *the product of the displacement by the component of the force in the direction of displacement*, that is,

$$\text{Work} = |\mathbf{S}|\,|\mathbf{F}| \cos\theta = \mathbf{F} \cdot \mathbf{S}, \tag{12}$$

where θ is the angle between $\mathbf{F}$ and $\mathbf{S}$.

If $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ is the resultant of two forces $\mathbf{F}_1$, $\mathbf{F}_2$ acting at the same point, the distributive law

$$\mathbf{F} \cdot \mathbf{S} = (\mathbf{F}_1 + \mathbf{F}_2) \cdot \mathbf{S} = \mathbf{F}_1 \cdot \mathbf{S} + \mathbf{F}_2 \cdot \mathbf{S}$$

expresses that the work of the resultant is equal to the sum of the works done by the separate forces.

When the point of application of a constant force $\mathbf{F}$ experiences two consecutive displacements $\mathbf{S}_1 = \overrightarrow{AB}$, $\mathbf{S}_2 = \overrightarrow{BC}$ (Fig. 15) the distributive law

$$\mathbf{F} \cdot \mathbf{S}_1 + \mathbf{F} \cdot \mathbf{S}_2 = \mathbf{F} \cdot (\mathbf{S}_1 + \mathbf{S}_2)$$

shows that the sum of the works in the two displacements is equal to the work done by $\mathbf{F}$ in the single displacement

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$$

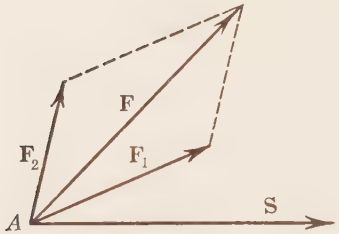


FIG. 14

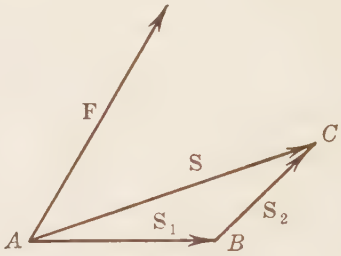


FIG. 15

from A to C. In a similar way we show that the total work done by a constant force when its point of application experiences any number of consecutive displacements, is equal to the work done by the force in the single displacement from the beginning to the end of motion. In particular, the total work done by a constant force is zero when its point of application traverses consecutively the sides of a closed polygon.

To a considerable extent the properties mentioned in the two preceding paragraphs are responsible for the importance of work in mechanics and physics, and for the importance of the scalar product in the study of mechanics and physics.

8. **Vector Product.** The **vector**, or **cross**, **product** of two vectors $\mathbf{A}$ and $\mathbf{B}$ (indicated by placing a small cross between the two factors) is defined as the vector

$$\mathbf{V} = \mathbf{A} \times \mathbf{B}$$

perpendicular to both $\mathbf{A}$ and $\mathbf{B}$, with magnitude

$$|\mathbf{V}| = |\mathbf{A}|\,|\mathbf{B}| \sin \theta, \tag{13}$$

and so directed that a right-handed rotation about $\mathbf{V}$ through an angle θ of not more than 180° carries $\mathbf{A}$ into $\mathbf{B}$.

Since

$$h = |\mathbf{B}| \sin \theta$$

is the altitude of the parallelogram with sides **A** and **B**, *the product* $\mathbf{A} \times \mathbf{B}$ *is equal in magnitude to the area of that parallelogram.*

The product $\mathbf{B} \times \mathbf{A}$ has the same magnitude as $\mathbf{A} \times \mathbf{B}$ but the rotation which carries **B** into **A** is opposite to that which carries **A** into **B**. Hence

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}. \qquad (14)$$

Thus *the vector product does not satisfy the commutative law.*

If **A** and **B** are parallel, θ is zero or 180° and

$$\mathbf{A} \times \mathbf{B} = 0.$$

Conversely, *if the vector product is zero, one of the vectors is zero or else the two are parallel.*

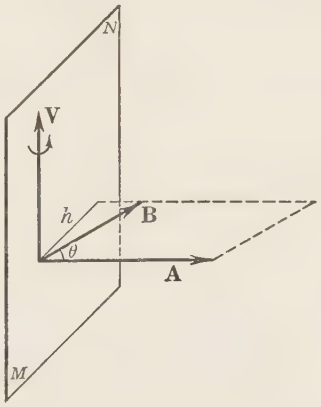


FIG. 16

The product $\mathbf{A} \times \mathbf{B}$ can be considered as the result obtained by projecting **B** on the plane MN perpendicular to **A**, rotating the projection 90° in the positive direction about **A**, and multiplying the resulting vector by $|\mathbf{A}|$. Each of these operations changes a closed polygon into a closed polygon. If they are

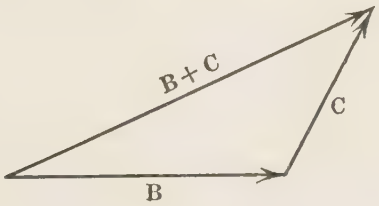


FIG. 17

performed on the sides **B**, **C**, $\mathbf{B} + \mathbf{C}$ of a triangle, the resulting vectors $\mathbf{A} \times \mathbf{B}$, $\mathbf{A} \times \mathbf{C}$, $\mathbf{A} \times (\mathbf{B} + \mathbf{C})$ then form the sides of a second triangle. Therefore

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}, \qquad (15)$$

whence

$$(\mathbf{B} + \mathbf{C}) \times \mathbf{A} = \mathbf{B} \times \mathbf{A} + \mathbf{C} \times \mathbf{A}, \qquad (16)$$

since each term of the latter equation merely differs in algebraic sign from the corresponding term of the former. These equations express the distributive law of vector multiplication.

In case of the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the definition of vector product gives immediately

$$\begin{aligned} \mathbf{i}\times\mathbf{i} &= \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0, \\ \mathbf{i}\times\mathbf{j} &= \mathbf{k} = -\mathbf{j}\times\mathbf{i}, \\ \mathbf{j}\times\mathbf{k} &= \mathbf{i} = -\mathbf{k}\times\mathbf{j}, \\ \mathbf{k}\times\mathbf{i} &= \mathbf{j} = -\mathbf{i}\times\mathbf{k}. \end{aligned} \tag{17}$$

By using these relations the cross product of two vectors can be expressed in terms of their components. Thus if

$$\begin{aligned} \mathbf{A} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \\ \mathbf{B} &= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \end{aligned}$$

by expanding and using (17) we find

$$\mathbf{A}\times\mathbf{B} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}. \tag{18}$$

As a determinant this can be written

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

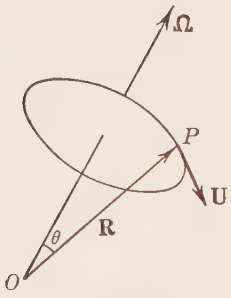


FIG. 18

9. Rotation About a Fixed Axis. Suppose a rigid body rotates with angular speed ω about an axis through O (Fig. 18). Its **angular velocity** is defined as the vector $\boldsymbol{\Omega}$ of magnitude ω which extends along the axis in the direction a right-handed screw advances when subject to the given rotation. Let $\mathbf{R}$ be the vector from O to a point P in the body. This point moves in a circle of radius $|\mathbf{R}|\sin\theta$ with speed $\omega\,|\mathbf{R}|\sin\theta$. Its velocity is therefore

$$\mathbf{U} = \boldsymbol{\Omega}\times\mathbf{R}. \tag{19}$$

For this vector has the magnitude

$$|\mathbf{U}| = |\boldsymbol{\Omega}|\,|\mathbf{R}|\sin\theta = \omega\,|\mathbf{R}|\sin\theta$$

of the velocity, and the definition of vector product indicates that it has the correct direction.

The point O from which $\mathbf{R}$ is drawn may be any point on the axis of rotation. If Ω_1 and Ω_2 are angular velocities of the same body about different axes, the point P will in the two rotations have velocities

$$\mathbf{U}_1 = \Omega_1 \times \mathbf{R}_1, \quad \mathbf{U}_2 = \Omega_2 \times \mathbf{R}_2,$$

where $\mathbf{R}_1$, $\mathbf{R}_2$ are vectors from points O_1, O_2 on the two axes to P. If the two axes intersect in a point O, we can replace $\mathbf{R}_1$, $\mathbf{R}_2$ by the vector $\mathbf{R}$ from O to P. Then

$$\mathbf{U}_1 = \Omega_1 \times \mathbf{R}, \quad \mathbf{U}_2 = \Omega_2 \times \mathbf{R}.$$

The distributive law

$$\Omega_1 \times \mathbf{R} + \Omega_2 \times \mathbf{R} = (\Omega_1 + \Omega_2) \times \mathbf{R}$$

thus expresses that the sum of the velocities experienced by the same point in two rotations Ω_1, Ω_2 about axes intersecting in a point, is equal to the velocity it has in the single rotation

$$\Omega = \Omega_1 + \Omega_2$$

about an axis through the same point. This is the principle of composition of angular velocities.

10. Moment of a Force. Let $\mathbf{F}$ be a force applied to a body at a point P. If $\mathbf{R}$ is the vector from any point O to P, the quantity

$$\mathbf{M} = \mathbf{R} \times \mathbf{F} \tag{20}$$

is called the **moment of F about the point** O.

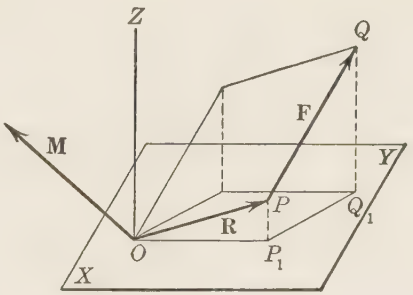


FIG. 19

The force tends to rotate the body about an axis through O perpendicular to the plane of O and $\mathbf{F}$. The moment is a vector extending along that axis in the direction a right-handed screw advances under such a rotation. In magnitude it is equal to the product of the force by its lever arm, or perpendicular distance from the axis.

Moment is usually defined with respect to an axis and not with respect to a point. The **moment of F about the axis** OZ is defined as the product of its component P_1Q_1 in a plane XY perpendicular to OZ by the distance from O to the line P_1Q_1. But that, being the area of the parallelogram OP_1Q_1, is also the projection of OPQ on XY. Now the parallelogram OPQ has an area equal to $|\mathbf{M}|$ and makes with XY an angle equal to that between $\mathbf{M}$ and OZ. Hence the projection of the parallelogram on XY is equal to the projection of $\mathbf{M}$ on OZ. Thus *the moment of* $\mathbf{F}$ *about any axis through* O *is equal to the component of* $\mathbf{M}$ *along that axis.*

From the definition it is clear that the moment of a force is not changed by moving the point of application along the line of action of the force. Also the distributive law

$$\mathbf{R} \times \mathbf{F}_1 + \mathbf{R} \times \mathbf{F}_2 = \mathbf{R} \times (\mathbf{F}_1 + \mathbf{F}_2)$$

shows that the sum of the moments of two forces acting at the same point is equal to the moment of their resultant.

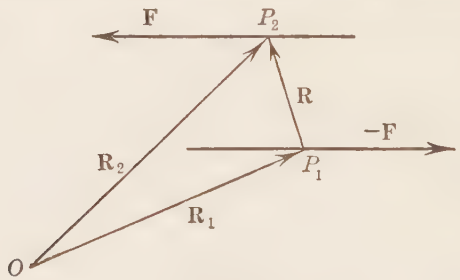


FIG. 20

A pair of forces $\mathbf{F}$, $-\mathbf{F}$, equal in magnitude, opposite in direction, and acting along parallel lines, is called a **couple.** If P_1, P_2 are points on the lines of action of $-\mathbf{F}$, $\mathbf{F}$ respectively, the moment of the couple about a point O is

$$\mathbf{R}_2 \times \mathbf{F} - \mathbf{R}_1 \times \mathbf{F} = (\mathbf{R}_2 - \mathbf{R}_1) \times \mathbf{F},$$

where $\mathbf{R}_1$, $\mathbf{R}_2$ are the vectors from O to P_1, P_2. Since

$$\mathbf{R}_2 - \mathbf{R}_1 = \mathbf{R}$$

is the vector from P_1 to P_2, the couple has the same moment

$$\mathbf{M} = \mathbf{R} \times \mathbf{F} \tag{21}$$

about every point.

11. Products Involving Three or More Vectors. By scalar and vector multiplication of three vectors **A**, **B**, **C**, triple products of the following types can be formed:

$$(1)\quad (\mathbf{A}\cdot\mathbf{B})\mathbf{C},$$
$$(2)\quad \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}),$$
$$(3)\quad \mathbf{A}\times(\mathbf{B}\times\mathbf{C}).$$

In these expressions parentheses have been introduced to indicate the order in which the multiplications are performed. Thus to obtain the quantity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$$

we first multiply **B** and **C** together and then multiply the result by **A**. In general

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$$

does not have the same value. Since the product **BC** without a dot or cross has not been defined, the expression $\mathbf{A}\cdot\mathbf{BC}$ can only mean $(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$. Also, since both factors of a cross product must be vectors, $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$ can only mean $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$. In products of types (1) and (2) the parentheses therefore may be omitted. For added clearness they are however often inserted.

The product $(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$ is merely the product of the scalar $\mathbf{A}\cdot\mathbf{B}$ and the vector **C**. This type of product was discussed in §3.

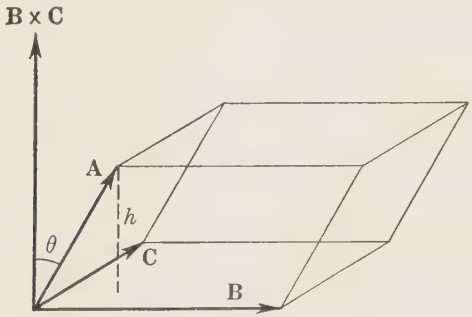


FIG. 21

The product $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$, *sometimes written* (ABC) *or* [ABC], *is a scalar equal in magnitude to the volume of the parallelepiped (Fig. 21) of which* **A**, **B**, **C** *are coterminous edges.* This is easily seen since $\mathbf{B}\times\mathbf{C}$ is a vector perpendicular to the base, of magni-

tude equal to the area of the base, and the projection of **A** on this vector is the altitude h of the parallelepiped. Hence

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = h\,|\mathbf{B} \times \mathbf{C}|$$

is equal to the volume.

The product $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ is positive when the angle θ is less than 90°, that is, when **A** and $\mathbf{B} \times \mathbf{C}$ lie on the same side of the plane through **B** and **C**. If **A** and $\mathbf{B} \times \mathbf{C}$ lie on opposite sides of that plane, the product is negative. If all three vectors are parallel to a plane, the volume of the parallelepiped is zero and

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = 0.$$

Inspection of a diagram will show that (**ABC**), (**BCA**), and (**CAB**) have the same algebraic sign. Since they are all in magnitude equal to the same volume, they are then equal. That is,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}). \tag{22}$$

The product is thus not changed by a cyclical permutation of the letters **A**, **B**, **C**. The other three products

$$\mathbf{A} \cdot (\mathbf{C} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{B} \times \mathbf{A}) \tag{23}$$

are easily seen to have the opposite algebraic sign.

Since

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}),$$

equation (22) shows that

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C}.$$

In the triple scalar product the dot and cross can therefore be interchanged without changing the value of the result.

If

$$\begin{aligned} \mathbf{A} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \\ \mathbf{B} &= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \\ \mathbf{C} &= c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}, \end{aligned}$$

the triple scalar product is equal to the determinant

$$(\mathbf{ABC}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

The product

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$$

is a vector sometimes encountered in analytical work involving repeated multiplication. It is evidently perpendicular to $\mathbf{B} \times \mathbf{C}$ (and so parallel to the plane of B and C) and also perpendicular to A. It is thus a vector parallel to the plane of B and C and perpendicular to A.

This product can be reduced to simpler form by means of the equation

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}. \tag{24}$$

To prove this we could express A, B, C in terms of the unit vectors i, j, k, expand both sides and show that the two are equal. We can greatly shorten the work, however, by using the fact that each term of the equation is a linear distributive function of each of the vectors. If the equation is satisfied by $\mathbf{A} = \mathbf{A}'$ and $\mathbf{A} = \mathbf{A}''$ and m', m'' are scalars, it is thus satisfied by

$$\mathbf{A} = m'\mathbf{A}' + m''\mathbf{A}''$$

and similarly for B and C. If then the equation is valid for every combination of the unit vectors, it is valid for all vectors. Now if each of the vectors A, B, C in (24) has one of the values i, j, k, both sides are zero unless B and C are different, and both sides are then zero unless A is equal to one of them. Since the relations between a pair of perpendicular unit vectors are the same for every pair, the equation will therefore be always satisfied if it is satisfied by the values $\mathbf{B} = \mathbf{i}$, $\mathbf{C} = \mathbf{j}$, $\mathbf{A} = \mathbf{i}$ or j. That it holds in these cases can be shown by direct substitution.

The product $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ is equivalent to $-\mathbf{C} \times (\mathbf{A} \times \mathbf{B})$. Expanding this by (24) and rearranging terms we have

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}. \tag{25}$$

The order of terms on the right of (24) and (25) can be remembered by noting that the first vector, B in both cases, is taken from the middle of the product on the left. This is sometimes called the *rule of the middle factor*.

By the use of these identities products containing four factors can be reduced to simpler form. Thus

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D})$$

can be considered either as the product of $\mathbf{A} \times \mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ or as the product of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C} \times \mathbf{D}$. Consequently,

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) &= (\mathbf{ABD})\mathbf{C} - (\mathbf{ABC})\mathbf{D} \\ &= (\mathbf{ACD})\mathbf{B} - (\mathbf{BCD})\mathbf{A}. \end{aligned} \tag{26}$$

Similarly $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D})$ can be considered as the scalar triple product of the vectors $\mathbf{A} \times \mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$. This is equivalent to

$$\begin{aligned}[(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}] \cdot \mathbf{D} &= [(\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}] \cdot \mathbf{D} \\ &= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D}).\end{aligned}$$

That is,

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}). \tag{27}$$

12. Functions of a Single Scalar Variable. Derivative. If to each value of a scalar variable t there corresponds a value of a vector $\mathbf{F}$, then $\mathbf{F}$ is called a function of t, a relation indicated by the notation $\mathbf{F}(t)$.

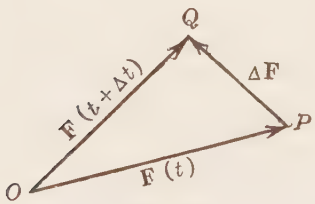


FIG. 22

When t approaches t_0, the variable vector $\mathbf{F}(t)$ is said to approach the constant vector $\mathbf{A}$ as limit if the magnitude of the difference $\mathbf{F}(t) - \mathbf{A}$ approaches zero. If $\mathbf{A}$ is not zero, this means that when t is near t_0, $\mathbf{F}(t)$ has almost the direction as well as almost the magnitude of $\mathbf{A}$. But if $\mathbf{A} = 0$ the direction of $\mathbf{F}(t)$ may vary arbitrarily provided merely that the magnitude $|\mathbf{F}(t)|$ approaches zero.

In Fig. 22, let

$$\mathbf{F}(t) = \overrightarrow{OP},$$

$$\mathbf{F}(t + \Delta t) = \overrightarrow{OQ}.$$

The increment in $\mathbf{F}(t)$ when t changes to $t + \Delta t$ is

$$\Delta\mathbf{F} = \mathbf{F}(t + \Delta t) - \mathbf{F}(t) = \overrightarrow{PQ}.$$

Since Δt is a scalar, the quotient

$$\frac{\Delta\mathbf{F}}{\Delta t} = \frac{\overrightarrow{PQ}}{\Delta t}$$

is a vector extending along PQ. If this vector approaches a limit when Δt approaches zero, that limit, written

$$\frac{d\mathbf{F}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{F}}{\Delta t}, \tag{28}$$

is called the **derivative** of $\mathbf{F}$ with respect to t. In particular, if t is the time $\frac{d\mathbf{F}}{dt}$ is called the rate of change of $\mathbf{F}$.

Let

$$\mathbf{R} = \overrightarrow{OP}$$

be the vector from a fixed point O to a point P moving along a curve (Fig. 23) and s the distance along the curve from a fixed point P_0 to P. When P moves to Q the increment $\Delta\mathbf{R}$ is the vector from P to Q. The ratio

$$\frac{\Delta \mathbf{R}}{\Delta s}$$

is a vector along the chord PQ of length

$$\left|\frac{\Delta \mathbf{R}}{\Delta s}\right| = \frac{\text{chord}}{\text{arc}}.$$

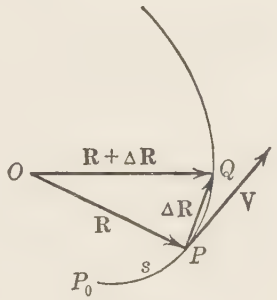


FIG. 23

As Q approaches P the line PQ approaches the tangent and the chord approaches the arc. Hence the limit

$$\frac{d\mathbf{R}}{ds}$$

is a vector of unit length along the tangent through P.

If t is the time,

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} \tag{29}$$

is called the **velocity** of P. Since

$$\frac{d\mathbf{R}}{dt} = \frac{d\mathbf{R}}{ds}\frac{ds}{dt},$$

it is a vector along the tangent of magnitude

$$|\mathbf{V}| = \frac{ds}{dt}.$$

The vector

$$\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d^2\mathbf{R}}{dt^2} \tag{30}$$

is called the **acceleration** of the moving point.

Derivatives of sums and products can be obtained by the use of formulas very much like those of calculus, the only difference being that *the order of factors must be preserved in each term of a product in case order has significance.* Thus suppose **A**, **B**, **C** are functions of the time and

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}.$$

If **A**, **B**, **C** receive increments $\Delta\mathbf{A}$, $\Delta\mathbf{B}$, $\Delta\mathbf{C}$, this becomes

$$\begin{aligned}\mathbf{C} + \Delta\mathbf{C} &= (\mathbf{A} + \Delta\mathbf{A}) \times (\mathbf{B} + \Delta\mathbf{B}) \\ &= \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \Delta\mathbf{B} + \Delta\mathbf{A} \times (\mathbf{B} + \Delta\mathbf{B}).\end{aligned}$$

By subtraction we find

$$\Delta\mathbf{C} = \mathbf{A} \times \Delta\mathbf{B} + \Delta\mathbf{A} \times (\mathbf{B} + \Delta\mathbf{B}),$$

whence

$$\frac{\Delta\mathbf{C}}{\Delta t} = \mathbf{A} \times \frac{\Delta\mathbf{B}}{\Delta t} + \frac{\Delta\mathbf{A}}{\Delta t} \times (\mathbf{B} + \Delta\mathbf{B}),$$

and in the limit

$$\frac{d\mathbf{C}}{dt} = \mathbf{A} \times \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \times \mathbf{B}.$$

In both terms on the right **A**, **B** occupy the same positions as in

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}.$$

It would not be correct to replace the second term by

$$\mathbf{B} \times \frac{d\mathbf{A}}{dt}.$$

13. Linear Vector Differential Equations. Linear differential equations containing one or more variable vectors and their derivatives with respect to scalars can be solved like scalar differential equations, the principal difference being that *the constants of integration are vectors.*

Thus, if the acceleration of a moving particle,

$$\frac{d^2\mathbf{R}}{dt^2} = \mathbf{A},$$

is constant, by direct integration we find

$$\frac{d\mathbf{R}}{dt} = \mathbf{V} = \mathbf{A}t + \mathbf{V}_0,$$

where $\mathbf{V}_0$ is a vector constant. By substituting $t = 0$ we show that this constant is the initial velocity. A second integration gives

$$\mathbf{R} = \tfrac{1}{2}\mathbf{A}t^2 + \mathbf{V}_0 t + \mathbf{R}_0,$$

where $\mathbf{R}_0$ is the value of $\mathbf{R}$ when $t = 0$.

As a second example, suppose the vectors $\mathbf{X}$, $\mathbf{Y}$ are functions of t which satisfy the equations

$$\frac{d\mathbf{X}}{dt} = k\mathbf{Y}, \quad \frac{d\mathbf{Y}}{dt} = -k\mathbf{X},$$

where k is a scalar constant. Differentiating the first equation and replacing $\frac{d\mathbf{Y}}{dt}$ from the second, we obtain

$$\frac{d^2\mathbf{X}}{dt^2} + k^2\mathbf{X} = 0.$$

The solution of this equation is

$$\mathbf{X} = \mathbf{A} \sin kt + \mathbf{B} \cos kt,$$

where $\mathbf{A}$ and $\mathbf{B}$ are vector constants. Substituting this value in the first equation we find

$$\mathbf{Y} = \mathbf{A} \cos kt - \mathbf{B} \sin kt.$$

14. Dynamics of a Particle or System of Particles. If m is the mass and $\mathbf{V}$ the velocity of a moving particle, the vector

$$m\mathbf{V}$$

is called the **momentum** of the particle. Newton's second law of motion is expressed by the equation

$$\frac{d}{dt}(m\mathbf{V}) = \mathbf{F}, \tag{31}$$

where $\mathbf{F}$ is the force acting on the particle. In mechanics the mass is usually constant and so this equation is equivalent to

$$m\frac{d\mathbf{V}}{dt} = \mathbf{F}, \tag{32}$$

where $\frac{d\mathbf{V}}{dt}$ is the acceleration of the particle.

If $\mathbf{R}$ is the vector from a fixed point O to the moving particle, the moment of momentum, or **angular momentum**, about O is defined as the vector

$$\mathbf{H} = \mathbf{R} \times m\mathbf{V} = m\mathbf{R} \times \mathbf{V}. \tag{33}$$

Since $$\mathbf{V} = \frac{d\mathbf{R}}{dt} \quad \text{and} \quad \mathbf{V} \times \mathbf{V} = 0,$$

$$\frac{d}{dt}(\mathbf{R} \times m\mathbf{V}) = \mathbf{R} \times \frac{d}{dt}(m\mathbf{V}) + \frac{d\mathbf{R}}{dt} \times m\mathbf{V} = \mathbf{R} \times \mathbf{F} = \mathbf{M},$$

where $\mathbf{M}$ is the moment, or torque, of the force $\mathbf{F}$ about O. The rate of change of angular momentum is thus

$$\frac{d\mathbf{H}}{dt} = \mathbf{M}. \tag{34}$$

Next consider the case of a system of particles of masses $m_1, m_2, \ldots, m_n$, moving with velocities $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n$, and subject to forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$, respectively. There is an equation of type (31) for each particle. Adding these together we get

$$\frac{d}{dt}(m_1\mathbf{V}_1 + m_2\mathbf{V}_2 + \cdots + m_n\mathbf{V}_n) = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n. \tag{35}$$

The total force

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n$$

is the sum of all the forces acting on the particles. These forces are of two kinds, namely, external forces exerted upon the particles by bodies outside the system, and internal forces exerted by the particles on each other. By Newton's third law, the forces exerted by two particles on each other are equal in magnitude and oppositely directed. In the sum each internal force thus occurs twice with opposite signs. *The internal forces therefore cancel, leaving* $\mathbf{F}$ *as the resultant of external forces.*

Let $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n$ be the vectors from a fixed point O to the particles $m_1, m_2, \ldots, m_n$ respectively. If the vector

$$\mathbf{R}_c = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2 + \cdots + m_n\mathbf{R}_n}{m_1 + m_2 + \cdots + m_n} \tag{36}$$

is drawn from the same point O, its end is a point C called the **center of gravity** of the system. Clearing of fractions, we have

$$m_1\mathbf{R}_1 + m_2\mathbf{R}_2 + \cdots + m_n\mathbf{R}_n = m\mathbf{R}_c,$$

where

$$m = m_1 + m_2 + \cdots + m_n$$

is the total mass of the system. Differentiation with respect to the time gives

$$m_1\mathbf{V}_1 + m_2\mathbf{V}_2 + \cdots + m_n\mathbf{V}_n = m\frac{d\mathbf{R}_c}{dt} = m\mathbf{V}_c,$$

where $\mathbf{V}_c$ is the velocity of the center of gravity. Equation (35) is thus equivalent to

$$\frac{d}{dt}(m\mathbf{V}_c) = \mathbf{F}. \tag{37}$$

Comparing this with (31) we see that *the center of gravity moves like a particle of mass equal to the total mass of the system under a force equal to the sum of all the external forces acting on the system.*

For each particle of the system there is an equation of type (34). Adding these together, we get

$$\frac{d\mathbf{H}}{dt} = \mathbf{M}, \tag{38}$$

where

$$\mathbf{H} = m_1\mathbf{R}_1 \times \mathbf{V}_1 + m_2\mathbf{R}_2 \times \mathbf{V}_2 + \cdots + m_n\mathbf{R}_n \times \mathbf{V}_n$$

is the total angular momentum about O of all the particles and

$$\mathbf{M} = \mathbf{R}_1 \times \mathbf{F}_1 + \mathbf{R}_2 \times \mathbf{F}_2 + \cdots + \mathbf{R}_n \times \mathbf{F}_n$$

is the total torque about O of all the forces. Since the internal forces exerted by a pair of particles on each other act along their

joining line, are equal in magnitude and opposite in direction, the torques due to these internal forces cancel in the sum, leaving $\mathbf{M}$ as the total torque due to external forces. Thus equation (38) expresses that *the rate of change of total angular momentum about any fixed point O is equal to the total moment of the external forces about that point.*

With respect to the moving center of gravity, the particle m_1 has velocity

$$\mathbf{V}_1 - \mathbf{V}_c$$

and angular momentum

$$m_1(\mathbf{R}_1 - \mathbf{R}_c) \times (\mathbf{V}_1 - \mathbf{V}_c).$$

The total angular momentum with respect to the center of gravity is therefore

$$\mathbf{H}_c = \sum m_i(\mathbf{R}_i - \mathbf{R}_c) \times (\mathbf{V}_i - \mathbf{V}_c).$$

Expanding and using the above relations this becomes

$$\mathbf{H}_c = \mathbf{H} - m\mathbf{R}_c \times \mathbf{V}_c,$$

whence

$$\mathbf{H} = \mathbf{H}_c + m\mathbf{R}_c \times \mathbf{V}_c. \tag{39}$$

Thus the angular momentum of a system of particles about a fixed point O is the sum of two parts, the first being their angular momentum about the center of gravity, the second the angular momentum about O of a particle equal to their total mass moving with the center of gravity.

Similarly, the moment of the forces about the center of gravity is

$$\mathbf{M}_c = \sum(\mathbf{R}_i - \mathbf{R}_c) \times \mathbf{F}_i = \mathbf{M} - \mathbf{R}_c \times \mathbf{F}$$

and so

$$\mathbf{M} = \mathbf{M}_c + \mathbf{R}_c \times \mathbf{F}. \tag{40}$$

The total moment of the forces about O is thus equal to their moment about the center of gravity plus the moment about O of a force equal to their resultant acting through the center of gravity.

By differentiating both sides of (39) with respect to the time and using equations (37), (38), and (40), we obtain

$$\frac{d\mathbf{H}_c}{dt} = \mathbf{M}_c. \tag{41}$$

This shows that the rate of change of angular momentum is equal to the total torque not only when these quantities are taken with respect to a fixed point O but also when taken with respect to the moving center of gravity.

PROBLEMS

1. $\mathbf{A}$, $\mathbf{B}$ are vectors forming consecutive sides of a parallelogram. Find the vectors forming the other two sides.

2. $\mathbf{A}$, $\mathbf{B}$ are vectors forming consecutive sides of a regular hexagon. Find the vectors forming the other four sides.

3. $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are vectors from the origin to the points A, B, C, D. If

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D},$$

show that $ABCD$ is a parallelogram.

4. Prove that

$$|\mathbf{A} + \mathbf{B}| \leqq |\mathbf{A}| + |\mathbf{B}|,$$
$$|\mathbf{A} - \mathbf{B}| \geqq |\mathbf{A}| - |\mathbf{B}|.$$

5. The vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ form coterminous edges of a parallelepiped. Show that

$$\mathbf{A} + \mathbf{B} + \mathbf{C}$$

is equal to a vector diagonal.

6. If $\mathbf{A}$, $\mathbf{B}$ are vectors from an origin O to the points A, B, find the vector from O to the middle point of AB.

7. A, B, C, D are four points, M_1 the middle point of AB, M_2 the middle point of CD, and M the middle point of M_1M_2. Show that the position of M is not affected by any interchange of the points A, B, C, D.

8. From the center of a regular pentagon vectors are drawn to its vertices. Show that their sum is zero.

9. $ABCD$ is a parallelogram. Lines are drawn from A to the middle points of BC and CD. Show that they trisect the diagonal BD.

10. If three numbers x, y, z (not all zero) exist such that

$$x\mathbf{A} + y\mathbf{B} + z\mathbf{C} = 0,$$

show that the vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are parallel to a plane.

11. $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are vectors extending from an origin O. If three numbers x, y, z (not all zero) can be found such that

$$x\mathbf{A} + y\mathbf{B} + z\mathbf{C} = 0,$$
$$x + y + z = 0,$$

show that the ends of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ lie in a line.

12. $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ are vectors extending from an origin O. If four numbers x_1, x_2, x_3, x_4 (not all zero) can be found such that

$$x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + x_3\mathbf{A}_3 + x_4\mathbf{A}_4 = 0,$$
$$x_1 + x_2 + x_3 + x_4 = 0,$$

show that the ends of $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ lie in a plane.

13. The vectors

$$\mathbf{A} = \mathbf{i} + \mathbf{j} - \mathbf{k},$$
$$\mathbf{B} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

extend from the origin of coördinates. Show that the line joining their ends is parallel to the xy-plane and find its length.

14. The vectors from the origin to the points A, B, C, D are

$$\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k},$$
$$\mathbf{B} = 2\mathbf{i} + 3\mathbf{j},$$
$$\mathbf{C} = 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k},$$
$$\mathbf{D} = \mathbf{k} - \mathbf{j}.$$

Show that the lines AB and CD are parallel and find the ratio of their lengths.

15. Show that the vectors $\mathbf{i} - \mathbf{j}$, $\mathbf{j} - \mathbf{k}$, $\mathbf{k} - \mathbf{i}$ are parallel to a plane.

16. Express the vector from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

17. Show that the vectors

$$\mathbf{A} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k},$$
$$\mathbf{B} = 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$

are perpendicular.

18. Show that the vectors

$$\mathbf{A} = 2\mathbf{i} - \mathbf{j} + \mathbf{k},$$
$$\mathbf{B} = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k},$$
$$\mathbf{C} = 3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$$

form the sides of a right triangle.

19. Find the cosine of the angle between the two vectors

$$\mathbf{A} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k},$$
$$\mathbf{B} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}.$$

20. If $\mathbf{R}_1$, $\mathbf{R}_2$ are vectors and r_1, r_2 their lengths, show that

$$\left(\frac{\mathbf{R}_2}{r_2^2} - \frac{\mathbf{R}_1}{r_1^2}\right)^2 = \frac{(\mathbf{R}_2 - \mathbf{R}_1)^2}{r_1^2 r_2^2}.$$

21. Prove that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its four sides.

22. By squaring both sides of the equation

$$\mathbf{A} = \mathbf{B} - \mathbf{C}$$

and interpreting the result geometrically, prove the formula

$$a^2 = b^2 + c^2 - 2bc \cos \alpha,$$

expressing the square of one side of a triangle in terms of the other two sides and their included angle.

23. Show that

$$\mathbf{A} = \mathbf{i} \cos \alpha + \mathbf{j} \sin \alpha,$$
$$\mathbf{B} = \mathbf{i} \cos \beta + \mathbf{j} \sin \beta$$

are unit vectors in the xy-plane making angles α, β with the x-axis. By means of the scalar product obtain the formula for $\cos(\alpha - \beta)$.

24. If r is the length of a vector $\mathbf{R}$ and α, β, γ the angles it makes with OX, OY, OZ respectively, show that

$$\mathbf{R} = r(\mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma).$$

The quantities $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called direction cosines of $\mathbf{R}$. Find the angle θ between two vectors $\mathbf{R}_1$, $\mathbf{R}_2$ with direction cosines $\cos\alpha_1$, $\cos\beta_1$, $\cos\gamma_1$ and $\cos\alpha_2$, $\cos\beta_2$, $\cos\gamma_2$ respectively.

25. If $\mathbf{A}$ is a constant vector and $\mathbf{R}$ is the vector from the origin to the variable point $P(x, y, z)$, show that

$$(\mathbf{R} - \mathbf{A}) \cdot \mathbf{A} = 0$$

is the equation of a plane.

26. If $\mathbf{A}$ is a constant vector and $\mathbf{R}$ is the vector from the origin to $P(x, y, z)$, show that

$$(\mathbf{R} - \mathbf{A}) \cdot \mathbf{R} = 0$$

is the equation of a sphere.

27. Given

$$\mathbf{A} = \mathbf{i} - 2\mathbf{j}, \qquad \mathbf{B} = \mathbf{j} + \mathbf{k},$$

find the component of $\mathbf{A}$ along $\mathbf{B}$.

28. If $\mathbf{A}$ is expressed as the sum of two vectors $\mathbf{A}'$ and $\mathbf{A}''$, respectively along and perpendicular to $\mathbf{B}$, show that

$$\mathbf{A}' = \frac{(\mathbf{A} \cdot \mathbf{B})\mathbf{B}}{\mathbf{B}^2}, \qquad \mathbf{A}'' = \mathbf{A} - \frac{(\mathbf{A} \cdot \mathbf{B})\mathbf{B}}{\mathbf{B}^2}.$$

29. Given

$$\mathbf{A} = \tfrac{1}{7}(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}),$$
$$\mathbf{B} = \tfrac{1}{7}(3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}),$$
$$\mathbf{C} = \tfrac{1}{7}(6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}),$$

show that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are of unit length, mutually perpendicular, and that $\mathbf{A} \times \mathbf{B} = \mathbf{C}$.

30. Find the area of the parallelogram formed by the two vectors

$$\mathbf{A} = 3\mathbf{i} + 2\mathbf{j}, \qquad \mathbf{B} = 2\mathbf{j} - 4\mathbf{k}.$$

31. Find the area of the triangle formed by the points

$$P_1(1, 1, 1), \qquad P_2(1, 2, 3), \qquad P_3(2, 3, 1).$$

32. If $\mathbf{u}_1$ and $\mathbf{u}_2$ are vectors of unit length and θ is the angle between them, show that

$$\sin\tfrac{1}{2}\theta = \tfrac{1}{2}|\mathbf{u}_2 - \mathbf{u}_1|.$$

33. The vectors from the origin to the points A, B, C are

$$\mathbf{A} = \mathbf{i} + \mathbf{j} - 2\mathbf{k},$$
$$\mathbf{B} = 2\mathbf{i} - \mathbf{j} + \mathbf{k},$$
$$\mathbf{C} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}.$$

Find a vector $\mathbf{N}$ perpendicular to the plane ABC. By projecting $\mathbf{A}$ upon $\mathbf{N}$ find the distance between the origin and the plane.

34. In the preceding problem find the vector which is the projection of $\mathbf{A}$ upon the plane ABC.

35. Using the values in problem 33, find the vector from A perpendicular to the line BC and terminating on that line.

36. If

$$\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k},$$
$$\mathbf{B} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k},$$
$$\mathbf{C} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k},$$
$$\mathbf{D} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

are the vectors from the origin to the points A, B, C, D, find a vector $\mathbf{N}$ perpendicular to the lines AB and CD. By projecting the vector $\overrightarrow{AC}$ on $\mathbf{N}$ find the shortest distance between the lines AB and CD.

37. Show that the law of refraction of light passing from a medium of refractive index μ into one of index μ' is expressed by the equation

$$\mu\mathbf{n} \times \mathbf{u} = \mu'\mathbf{n} \times \mathbf{u}',$$

where $\mathbf{n}$, $\mathbf{u}$, $\mathbf{u}'$ are unit vectors perpendicular to the boundary, along the incident, and along the refracted rays respectively.

38. Find the value of $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ if

$$\mathbf{A} = \mathbf{i} - \mathbf{j} - 6\mathbf{k},$$
$$\mathbf{B} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k},$$
$$\mathbf{C} = 2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}.$$

39. If $\mathbf{i}_1$, $\mathbf{j}_1$, $\mathbf{k}_1$ are mutually perpendicular unit vectors and $\mathbf{k}_1 = \mathbf{i}_1 \times \mathbf{j}_1$, show that the vector

$$\mathbf{U} = \mathbf{i} \times \mathbf{i}_1 + \mathbf{j} \times \mathbf{j}_1 + \mathbf{k} \times \mathbf{k}_1$$

makes equal angles with $\mathbf{i}$ and $\mathbf{i}_1$, $\mathbf{j}$ and $\mathbf{j}_1$, $\mathbf{k}$ and $\mathbf{k}_1$. It thus extends along the axis of the rotation which carries $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ into $\mathbf{i}_1$, $\mathbf{j}_1$, $\mathbf{k}_1$.

40. If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are vectors from the origin to the points A, B, C, show that

$$\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}$$

is perpendicular to the plane ABC.

41. Prove the formula

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A}) = (\mathbf{ABC})^2.$$

42. Show that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{0}.$$

43. Show that

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} \times \mathbf{D} + \mathbf{B} \times \mathbf{C} \cdot \mathbf{A} \times \mathbf{D} + \mathbf{C} \times \mathbf{A} \cdot \mathbf{B} \times \mathbf{D} = 0.$$

44. By means of products express the condition that the plane through $\mathbf{A}$ and $\mathbf{B}$ be perpendicular to that through $\mathbf{C}$ and $\mathbf{D}$.

45. By means of products express the condition that **A**, **B**, **C**, **D** be all parallel to a plane.

46. If **F** is a function of t find the derivative of

$$\mathbf{F} \cdot \frac{d\mathbf{F}}{dt} \times \frac{d^2\mathbf{F}}{dt^2}.$$

47. If $\mathbf{F}(t)$ is a vector of constant magnitude, show that

$$\mathbf{F} \cdot \frac{d\mathbf{F}}{dt} = 0.$$

48. At time t the vector from the origin to a moving point is

$$\mathbf{R} = \mathbf{A} \cos \omega t + \mathbf{B} \sin \omega t,$$

where **A**, **B**, ω are constants.

(*a*) Find the velocity **V** and show that

$$\mathbf{R} \times \mathbf{V} = \text{constant}.$$

(*b*) Show that the acceleration is directed toward the origin and is proportional to the distance.

49. By differentiating both sides of the equation

$$\mathbf{V} = \frac{d\mathbf{R}}{ds} \frac{ds}{dt}$$

with respect to the time and interpreting the result, obtain the tangential and normal components of acceleration.

50. A particle of mass m is attracted toward the origin with a force k^2m times the distance. If it starts from $\mathbf{R} = \mathbf{R}_0$ with velocity $\mathbf{V}_0$, find its position at time t.

51. If the particle in the preceding problem is repelled with a force k^2m times the distance, find its position at time t.

52. If $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n$ are the vectors from a point O to particles of masses $m_1, m_2, \ldots, m_n$ respectively, **C** is the vector from O to their center of gravity, and **R** is any vector, show that

$$\sum_{i=1}^{n} m_i(\mathbf{R} - \mathbf{R}_i)^2 = m(\mathbf{R} - \mathbf{C})^2 + \sum_{i=1}^{n} m_i(\mathbf{C} - \mathbf{R}_i)^2,$$

where m is the total mass.

53. In case of a rigid body rotating about an axis of symmetry, show that the total angular momentum is a vector along that axis. From this fact and equation (38) show that if the axis is horizontal and supported at one end the force of gravity tends to make the axis rotate in a horizontal plane about the point of support. This is the principle of the gyroscope.

CHAPTER II

PARTIAL DIFFERENTIATION

15. Functions of Position. If to each point of a region there corresponds a value of some physical quantity, the region is called a **field.** It is called a scalar field when the quantity under consideration is scalar and a vector field when that quantity is a vector. In a moving fluid, for instance, temperature determines a scalar field while the velocity of the fluid defines a vector field. A scalar field may be visualized as filled with numbers, one attached to each point. Similarly, a vector field is filled with vectors, one issuing from each point.

If f is a scalar having a value at each point $P(x, y, z)$ of a region, we indicate this by the notation $f(x, y, z)$ or $f(P)$. If

$$\mathbf{R} = \overrightarrow{OP}$$

is the vector from the origin to P, then f is a function of $\mathbf{R}$ in the sense that to each value of $\mathbf{R}$ there corresponds a point P and so a value of f. We indicate this by writing

$$f = f(\mathbf{R}).$$

Similarly, if $\mathbf{F}$ is a vector having a value at each point $P(x, y, z)$ we indicate this through the notation $\mathbf{F}(x, y, z)$, $\mathbf{F}(P)$, or $\mathbf{F}(\mathbf{R})$.

These functions having values at each point are called point functions or **functions of position.** Unless it is otherwise expressly stated we shall understand that such functions are **uniform,** or one-valued. By this we mean that each function has one value at each point, though usually of course different values at different points.

The quantities of physics are usually functions of time as well as position. The values of such a quantity at different points at a particular time constitute an instantaneous field. It is this instantaneous field with which we shall be mainly concerned. Thus when we speak of the velocity in a moving fluid, or the electric or magnetic intensity in a region, we shall

usually mean the value of the quantity in question at a particular time.

16. Continuity.* A variable point P is said to approach a fixed point P_0 as limit if the distance P_0P has the limit zero. A function of position is called **continuous** at P_0 if it has a definite value at P_0 and approaches that value as limit whenever P approaches P_0 as limit.

The analytical conditions that $f(P)$ be continuous at P_0 are (1) that $f(P_0)$ have a definite value and (2) that to each positive number ϵ there corresponds a positive number δ such that

$$|f(P) - f(P_0)| < \epsilon \tag{42}$$

whenever

$$P_0P < \delta. \tag{43}$$

The condition of continuity for a vector function has the same form, **F** merely replacing f in these expressions.

A function is called continuous in a region if these conditions hold for all points P_0, P belonging to the region. In that case the values of δ for which (42) is satisfied usually depend on the point P_0. The function is called **uniformly continuous** in the region if to each value of ϵ there corresponds a value of δ, independent of P_0, such that (42) is valid for every pair of points P_0, P belonging to the region and separated by a distance less than δ.

The condition of continuity expresses that a sphere of center P_0 can be found within which the function nowhere differs from its value at the center by an amount in absolute value greater than ϵ. As P_0 varies the radius of the largest sphere within which this is true usually varies. But in a region where the continuity is uniform this largest radius never becomes less than some number δ_0. Within a sphere of radius δ_0 condition (42) is then satisfied for all points P belonging to the region, wherever the center P_0 lies in the region.

A function $f(P)$ is said to have a limit c at infinity if to each positive number ϵ there corresponds a sphere S outside which the condition

$$|f(P) - c| < \epsilon$$

* For reference we here give definitions of certain mathematical terms which will be occasionally used. The reader desiring further information about these matters is referred to E. B. Wilson's "Advanced Calculus," or O. D. Kellogg's "Potential Theory."

is everywhere satisfied. A function is called continuous at infinity if it has a limit at infinity.

In a given region a function is called **bounded** if there exists a number m such that the function is in absolute value less than m at every point of the region. A function uniformly continuous in a finite region is bounded.

A point P is called an **interior point** of a region V if there is a solid sphere of center P all points of which belong to V. If the points of a region are all interior points, the region is called **open.** If it contains all its limit points, the region is called **closed.** The points inside a sphere form an open region. These and the points on the surface of the sphere form a closed region.

This property of being closed derives its importance mainly from the fact that *a function continuous at all points of a closed region is uniformly continuous in that region.* In discussions where uniform continuity is required it is then sufficient to show that the function is continuous at all points of the region and its boundary.

A function is often continuous at interior points of a region but is not even defined at points of the boundary. We call a function $f(x, y, z)$ **continuously differentiable** in a region if at interior points the partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}$$

exist and there coincide with functions continuous in the region (including the boundary if the region is closed).

Consider a finite region V divided into a finite number of regular sub-regions. If a function is uniformly continuous at interior points of each sub-region, it is called **piecewise continuous** in V. If the function and its first derivatives with respect to x, y, z are uniformly continuous at interior points of the sub-regions, the function is called **piecewise differentiable** in V. In a sphere with center at the origin, the function which is equal to y when x is negative and equal to z when x is positive, is piecewise continuous and piecewise differentiable, but on the plane $x = 0$ is discontinuous.

17. Gradient of a Scalar Function. Let $\phi(x, y, z)$ be a scalar function with first derivatives continuous in a certain region, and

$$\mathbf{R} = \overrightarrow{OP} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

the vector from the origin O to the variable point $P(x, y, z)$. In a small variation of x, y, z,

$$d\mathbf{R} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz, \tag{44}$$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz. \tag{45}$$

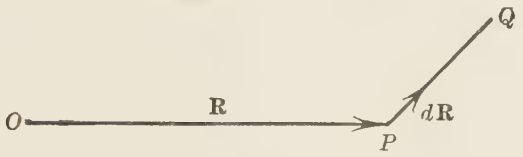


FIG. 24

If the variation is taken in the direction PQ and

$$ds = |d\mathbf{R}|$$

is the distance moved,

$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds} \tag{46}$$

is a quantity called *the derivative of ϕ in the direction PQ*. At a point P there is a derivative of ϕ in each direction. In particular if PQ has the direction of the x-axis, $dy = dz = 0$, $dx = ds$, and so

$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}.$$

The partial derivative with respect to one of the coördinates is thus merely the value of $\dfrac{d\phi}{ds}$ taken in the positive direction of the corresponding axis.

By direct multiplication,

$$\left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right)\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz) = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz.$$

Equation (45) is therefore equivalent to

$$d\phi = \left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right)\cdot d\mathbf{R}. \tag{47}$$

The quantity

$$\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} = \operatorname{grad}\phi \tag{48}$$

is a vector called the **gradient** of ϕ. To determine its significance write (47) in the form

$$d\phi = \text{grad } \phi \cdot d\mathbf{R}. \tag{49}$$

Then

$$\frac{d\phi}{ds} = \text{grad } \phi \cdot \frac{d\mathbf{R}}{ds} \tag{50}$$

is the derivative of ϕ in the direction $d\mathbf{R}$. Now $\frac{d\mathbf{R}}{ds}$ is a vector of unit length and so

$$\text{grad } \phi \cdot \frac{d\mathbf{R}}{ds}$$

is the component of grad ϕ in the direction $d\mathbf{R}$. Thus *grad ϕ is a vector which has for component in any direction the derivative of ϕ in that direction.*

At a given point P the component $\frac{d\phi}{ds}$ will have its greatest value when $d\mathbf{R}$ has the same direction as grad ϕ. Hence at each point of the region *grad ϕ extends in the direction in which the derivative of ϕ has its maximum value and is in magnitude equal to that maximum derivative.*

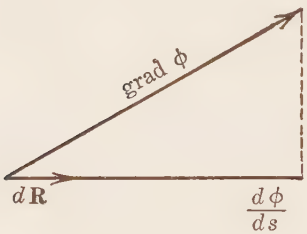


FIG. 25

These properties show that the gradient of ϕ depends only on the way the values of ϕ are distributed in space and is entirely independent of the position and direction of the coördinate axes. In the expression

$$\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$

for gradient we may therefore replace $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ by unit vectors along any three perpendicular axes and $\frac{\partial\phi}{\partial x}$, $\frac{\partial\phi}{\partial y}$, $\frac{\partial\phi}{\partial z}$ by the derivatives of ϕ in the direction of those axes. Thus, if $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are any three mutually perpendicular unit vectors and $\frac{d\phi}{ds_1}$, $\frac{d\phi}{ds_2}$, $\frac{d\phi}{ds_3}$ are the derivatives of ϕ in the direction of those vectors,

$$\text{grad } \phi = \mathbf{i}_1\frac{d\phi}{ds_1} + \mathbf{i}_2\frac{d\phi}{ds_2} + \mathbf{i}_3\frac{d\phi}{ds_3}. \tag{51}$$

Through each point $P(x, y, z)$ of the field there passes a surface

$$\phi(x, y, z) = \text{constant}.$$

In a variation along this surface,

$$d\phi = \text{grad}\,\phi \cdot d\mathbf{R} = 0.$$

This expresses that grad ϕ is perpendicular to every vector $\mathbf{dR}$ in the surface at P. Hence *grad ϕ is normal to the surface ϕ = constant at P.*

To show this in another way consider the system of surfaces

$$\phi(x, y, z) = \text{constant}$$

for various values of the constant. Designate each of these surfaces by the value of ϕ on it. On one side of the surface through P the function has larger values than at P, on the other side smaller values. In passing from P to any point Q of the surface $\phi + d\phi$, the increase in ϕ is the same whatever point Q on the latter surface is chosen. The distance PQ will be least and so the ratio $\dfrac{d\phi}{ds}$ greatest when PQ has the direction PN of the normal. Hence the vector grad ϕ is normal to the surface ϕ = constant through P and is directed toward the side on which ϕ increases.

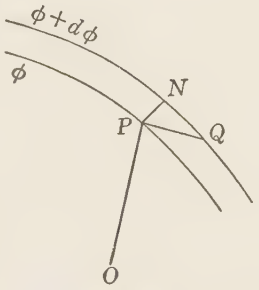


FIG. 26

18. The Operator ∇. The notation

$$\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi$$

is used to represent the sum

$$\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$

obtained by differentiating ϕ with respect to x, y, z, multiplying the derivatives respectively by i, j, k, and adding the results. This may be considered as obtained by applying to ϕ the vector differential operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \qquad (52)$$

This operator, called del, is one of the most important differential operators in physics. In terms of it the gradient of ϕ can be expressed in the form

$$\text{grad}\,\phi = \nabla\phi. \tag{53}$$

If $\mathbf{R}$ is the vector from the origin to the point $P(x, y, z)$, equation (49) shows that the total differential of ϕ has the form

$$d\phi = d\mathbf{R} \cdot \nabla\phi. \tag{54}$$

The derivative of ϕ in the direction $d\mathbf{R}$ is then

$$\frac{d\phi}{ds} = \frac{d\mathbf{R}}{ds} \cdot \nabla\phi, \tag{55}$$

and if $\mathbf{u}$ is any unit vector

$$\mathbf{u} \cdot \nabla\phi$$

is the derivative of ϕ in the direction $\mathbf{u}$.

The scalar product of $d\mathbf{R}$ and ∇ is the operator

$$d\mathbf{R} \cdot \nabla = dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}.$$

The total differential of a vector $\mathbf{F}(x, y, z)$ is then

$$d\mathbf{F} = \frac{\partial \mathbf{F}}{\partial x}dx + \frac{\partial \mathbf{F}}{\partial y}dy + \frac{\partial \mathbf{F}}{\partial z}dz = d\mathbf{R} \cdot \nabla\mathbf{F}.$$

The derivative of $\mathbf{F}$ in the direction $d\mathbf{R}$ is therefore

$$\frac{d\mathbf{F}}{ds} = \frac{d\mathbf{R}}{ds} \cdot \nabla\mathbf{F}, \tag{56}$$

and if $\mathbf{u}$ is any unit vector

$$\mathbf{u} \cdot \nabla\mathbf{F}$$

is the derivative of $\mathbf{F}$ in the direction $\mathbf{u}$. Similarly, if

$$\mathbf{U} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

is any vector

$$\mathbf{U} \cdot \nabla\mathbf{F} = \left(u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y} + u_3\frac{\partial}{\partial z}\right)\mathbf{F} \tag{57}$$

is the product of the magnitude of $\mathbf{U}$ and the derivative of $\mathbf{F}$ in the direction of $\mathbf{U}$.

Since ∇ and $\mathbf{F}$ are both vectors the combination $\nabla\mathbf{F}$ has not been defined. Hence $\mathbf{U}\cdot\nabla\mathbf{F}$ can only mean $\mathbf{U}\cdot\nabla$ operating on $\mathbf{F}$. Also it is understood that ∇ operates only on terms which stand on its right to the nearest parenthesis or bracket.* Hence $\mathbf{U}\cdot\nabla\phi$ and

$$(\nabla\cdot\mathbf{U})\phi = \phi\nabla\cdot\mathbf{U}$$

are in general not equal. Similarly $\mathbf{U}\cdot\nabla\mathbf{F}$ is in general not equivalent to $(\nabla\cdot\mathbf{U})\mathbf{F}$.

If ϕ_1 and ϕ_2 are scalar functions of x, y, z it is clear that

$$\nabla(\phi_1+\phi_2) = \nabla\phi_1 + \nabla\phi_2.$$

In case of a vector function $\mathbf{F}(x, y, z)$ we shall make extensive use of the quantities

$$\nabla\cdot\mathbf{F} = \mathbf{i}\cdot\frac{\partial\mathbf{F}}{\partial x} + \mathbf{j}\cdot\frac{\partial\mathbf{F}}{\partial y} + \mathbf{k}\cdot\frac{\partial\mathbf{F}}{\partial z},$$

$$\nabla\times\mathbf{F} = \mathbf{i}\times\frac{\partial\mathbf{F}}{\partial x} + \mathbf{j}\times\frac{\partial\mathbf{F}}{\partial y} + \mathbf{k}\times\frac{\partial\mathbf{F}}{\partial z}.$$

By expanding both sides it is easily seen that

$$\nabla\cdot(\mathbf{F}_1+\mathbf{F}_2) = \nabla\cdot\mathbf{F}_1 + \nabla\cdot\mathbf{F}_2,$$
$$\nabla\times(\mathbf{F}_1+\mathbf{F}_2) = \nabla\times\mathbf{F}_1 + \nabla\times\mathbf{F}_2.$$

All of these operations with ∇ are thus distributive.

Since it contains the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the operator ∇ appears to depend on the coördinate axes. This is however not the case. In fact if $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are any three mutually perpendicular unit vectors and $\frac{d}{ds_1}, \frac{d}{ds_2}, \frac{d}{ds_3}$ represent differentiations in the direction of those vectors,

$$\nabla = \mathbf{i}_1\frac{d}{ds_1} + \mathbf{i}_2\frac{d}{ds_2} + \mathbf{i}_3\frac{d}{ds_3}. \tag{58}$$

By this we mean that the operations represented by the right sides of (52) and (58) when performed on the same function give equal results. In case of a scalar function ϕ this follows from (51).

* See however footnote page 72.

In case of the quantities $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ formed by operating on a vector function

$$\mathbf{F} = \mathbf{i}f_1 + \mathbf{j}f_2 + \mathbf{k}f_3,$$

this is again true because the results can be written

$$\nabla \cdot \mathbf{F} = \mathbf{i} \cdot \nabla f_1 + \mathbf{j} \cdot \nabla f_2 + \mathbf{k} \cdot \nabla f_3,$$
$$\nabla \times \mathbf{F} = -\mathbf{i} \times \nabla f_1 - \mathbf{j} \times \nabla f_2 - \mathbf{k} \times \nabla f_3,$$

and these contain ∇ only in the gradients ∇f_1, ∇f_2, ∇f_3 which we have already shown to have the same values whether ∇ has the form (52) or (58).

When applied to composite functions operation with ∇ has many of the properties of a partial differentiation with respect to a coördinate. Thus, if u is a function of x, y, z,

$$\nabla f(u) = \mathbf{i}\frac{df}{du}\frac{\partial u}{\partial x} + \mathbf{j}\frac{df}{du}\frac{\partial u}{\partial y} + \mathbf{k}\frac{df}{du}\frac{\partial u}{\partial z} = \frac{df}{du}\nabla u.$$

Similarly, if $u_1, u_2, \ldots, u_n$ are functions of x, y, z and f is a function of $u_1, u_2, \ldots, u_n$,

$$\nabla f = \frac{\partial f}{\partial u_1}\nabla u_1 + \frac{\partial f}{\partial u_2}\nabla u_2 + \cdots + \frac{\partial f}{\partial u_n}\nabla u_n. \tag{59}$$

19. Differentiation Formulas. Let ϕ be a scalar and $\mathbf{U}, \mathbf{V}$ vector functions of x, y, z. By direct expansion the following identities can be proved:

$$\nabla \cdot \phi\mathbf{U} = \phi\nabla \cdot \mathbf{U} + \mathbf{U} \cdot \nabla\phi \tag{60}$$
$$\nabla \times \phi\mathbf{U} = \phi\nabla \times \mathbf{U} + \nabla\phi \times \mathbf{U} \tag{61}$$
$$\nabla \cdot \mathbf{U} \times \mathbf{V} = \mathbf{V} \cdot \nabla \times \mathbf{U} - \mathbf{U} \cdot \nabla \times \mathbf{V} \tag{62}$$
$$\nabla \times (\mathbf{U} \times \mathbf{V}) = \mathbf{V} \cdot \nabla\mathbf{U} - \mathbf{U} \cdot \nabla\mathbf{V} + \mathbf{U}(\nabla \cdot \mathbf{V}) - \mathbf{V}(\nabla \cdot \mathbf{U}) \tag{63}$$
$$\nabla(\mathbf{U} \cdot \mathbf{V}) = \mathbf{U} \cdot \nabla\mathbf{V} + \mathbf{V} \cdot \nabla\mathbf{U} + \mathbf{U} \times (\nabla \times \mathbf{V}) + \mathbf{V} \times (\nabla \times \mathbf{U}) \tag{64}$$
$$\nabla \times (\nabla\phi) = 0 \tag{65}$$
$$\nabla \cdot (\nabla \times \mathbf{U}) = 0 \tag{66}$$
$$\nabla \times (\nabla \times \mathbf{U}) = \nabla(\nabla \cdot \mathbf{U}) - \nabla^2\mathbf{U} \tag{67}$$

If $$\mathbf{R} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

is the vector from the origin to the point $P(x, y, z)$,

$$\nabla \cdot \mathbf{R} = 3 \tag{68}$$
$$\nabla \times \mathbf{R} = 0 \tag{69}$$
$$\mathbf{U} \cdot \nabla\mathbf{R} = \mathbf{U} \tag{70}$$

In these formulas ∇ has the character both of a vector and a differential operator. Some of the formulas are obtained from vector identities by replacing a vector by ∇. In an identity every term is cancelled by an exactly equal term occurring elsewhere. Since it is a distributive operator, ∇ may be substituted for a vector in such an identity provided it is restricted to operate on the same variables in all the terms. We usually assume that a derivative symbol operates on all terms that stand on its right (at least to the nearest parenthesis or bracket). If this is not the case we must use some notation to indicate the quantities differentiated. It is sometimes convenient to indicate the variable quantities by subscripts. Thus we shall use the notation

$$\nabla_U(\mathbf{U} \cdot \mathbf{V})$$

to indicate the gradient with $\mathbf{U}$ variable and $\mathbf{V}$ constant.*

As an example let $\mathbf{C}$ in the vector identity

$$\mathbf{U} \times (\mathbf{C} \times \mathbf{V}) = \mathbf{C}(\mathbf{U} \cdot \mathbf{V}) - (\mathbf{U} \cdot \mathbf{C})\mathbf{V}$$

be replaced by ∇_V. The result is

$$\mathbf{U} \times (\nabla \times \mathbf{V}) = \nabla_V(\mathbf{U} \cdot \mathbf{V}) - \mathbf{U} \cdot \nabla\mathbf{V}.$$

In the first and last terms the subscript is omitted because the order indicates correctly the variable on which ∇ operates. Similarly,

$$\mathbf{V} \times (\nabla \times \mathbf{U}) = \nabla_U(\mathbf{U} \cdot \mathbf{V}) - \mathbf{V} \cdot \nabla\mathbf{U}.$$

Combining these equations, we obtain

$$\begin{aligned}&\nabla_U(\mathbf{U} \cdot \mathbf{V}) + \nabla_V(\mathbf{U} \cdot \mathbf{V}) \\ &\quad = \mathbf{U} \cdot \nabla\mathbf{V} + \mathbf{V} \cdot \nabla\mathbf{U} + \mathbf{U} \times (\nabla \times \mathbf{V}) + \mathbf{V} \times (\nabla \times \mathbf{U}).\end{aligned}$$

If a function contains several variable quantities, its derivative is obtained by differentiating with one of them variable at a time and adding the results. Hence

$$\nabla(\mathbf{U} \cdot \mathbf{V}) = \nabla_U(\mathbf{U} \cdot \mathbf{V}) + \nabla_V(\mathbf{U} \cdot \mathbf{V})$$

and so the above equation is equivalent to (64).

* Subscripts are sometimes also used to indicate quantities which are constant. Compare, for example, ∇_t, page 201.

20. Functional Relations. In a certain region let

$$u = f_1(x, y, z),$$
$$v = f_2(x, y, z),$$
$$w = f_3(x, y, z)$$

be continuously differentiable functions of x, y, z.

First, suppose u and v satisfy an equation

$$f(u, v) = 0. \tag{71}$$

Differentiation gives (equation (59))

$$\frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v = 0.$$

Taking the vector product with ∇u we have

$$\frac{\partial f}{\partial v}\nabla u \times \nabla v = 0. \tag{72}$$

If u is constant ∇u and so also $\nabla u \times \nabla v$ is zero. If u is not constant, equation (71) must contain v. The partial derivative $\frac{\partial f}{\partial v}$ is then not identically zero and (72) requires

$$\nabla u \times \nabla v = 0. \tag{73}$$

Conversely, suppose u and v satisfy (73). If one of these functions, for example u, is constant, the equation

$$u - c = 0$$

is a functional relation of type (71). If neither is constant, consider any surface

$$u = \text{constant}.$$

At points of this surface equation (73) expresses that ∇v is parallel to ∇u and so has no component in the tangent plane. As we move along the surface, v therefore does not change. Consequently v is a function of u and so again u, v satisfy a functional relation.

We have therefore proved that

$$\nabla u \times \nabla v = 0$$

is the necessary and sufficient condition that u, v *satisfy an equation*

$$f(u, v) = 0.$$

Next suppose u, v, w satisfy an equation

$$f(u, v, w) = 0. \tag{74}$$

Taking the gradient and multiplying by $\nabla u \times \nabla v$ we obtain

$$\frac{\partial f}{\partial w} \nabla u \times \nabla v \cdot \nabla w = 0. \tag{75}$$

If u and v alone satisfy a functional relation, $\nabla u \times \nabla v = 0$ and so $\nabla u \times \nabla v \cdot \nabla w = 0$. If u and v do not satisfy such a relation, (74) must contain w. Then $\dfrac{\partial f}{\partial w}$ is not identically zero and (75) requires

$$\nabla u \times \nabla v \cdot \nabla w = \nabla u \cdot \nabla v \times \nabla w = 0. \tag{76}$$

Conversely, suppose u, v, w satisfy (76). If

$$\nabla u \times \nabla v = 0,$$

equation (71) which is then satisfied is a special case of (74). If $\nabla u \times \nabla v$ is not zero, consider any curve

$$u = \text{constant}, \quad v = \text{constant}.$$

At points of this curve ∇u and ∇v are perpendicular to the tangent. Equation (76) expresses that ∇u, ∇v, ∇w lie in a plane. Hence ∇w is also perpendicular to the tangent and so has no component along the curve. As we move along the curve, w therefore does not change. Consequently w is a function of u and v and so u, v, w satisfy a functional relation.

We have therefore proved that

$$\nabla u \cdot \nabla v \times \nabla w = 0$$

is the necessary and sufficient condition that u, v, w satisfy an equation

$$f(u, v, w) = 0.$$

The quantity

$$\nabla u \cdot \nabla v \times \nabla w = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} \tag{77}$$

is called the **Jacobian** of u, v, w with respect to x, y, z. It is usually represented by one of the notations

$$J\left(\frac{u, v, w}{x, y, z}\right) = \frac{\partial(u, v, w)}{\partial(x, y, z)}.$$

In the given region we have assumed that

$$u = f_1(x, y, z), \quad v = f_2(x, y, z), \quad w = f_3(x, y, z) \tag{78}$$

are continuously differentiable functions of x, y, z. With each point of the region are associated values of x, y, z and values of u, v, w. Suppose x, y, z are also continuously differentiable functions

$$x = \phi_1(u, v, w), \quad y = \phi_2(u, v, w), \quad z = \phi_3(u, v, w) \tag{79}$$

of u, v, w. We shall then have

$$\nabla x = \frac{\partial \phi_1}{\partial u}\nabla u + \frac{\partial \phi_1}{\partial v}\nabla v + \frac{\partial \phi_1}{\partial w}\nabla w,$$

$$\nabla y = \frac{\partial \phi_2}{\partial u}\nabla u + \frac{\partial \phi_2}{\partial v}\nabla v + \frac{\partial \phi_2}{\partial w}\nabla w,$$

$$\nabla z = \frac{\partial \phi_3}{\partial u}\nabla u + \frac{\partial \phi_3}{\partial v}\nabla v + \frac{\partial \phi_3}{\partial w}\nabla w.$$

Forming the triple scalar product of these three expressions we get on the left

$$\nabla x \cdot \nabla y \times \nabla z = \mathbf{i} \cdot \mathbf{j} \times \mathbf{k} = 1,$$

and on the right a sum of 27 terms. Each of these terms is zero which contains the same factor ∇u, ∇v, ∇w more than once and each of the remaining terms contains the Jacobian $\nabla u \cdot \nabla v \times \nabla w$ as factor. If the Jacobian were zero we should then have

$$\nabla x \cdot \nabla y \times \nabla z = 0$$

which is impossible.

We have therefore proved that if in a given region equations (78) can be solved for x, y, z as continuously differentiable functions of u, v, w the Jacobian

$$\nabla u \cdot \nabla v \times \nabla w$$

cannot vanish at any point of the region. In books on Advanced Calculus* it is proved conversely that if the Jacobian is not

* cf. F. S. Woods, "Advanced Calculus," page 101.

zero equations (78) can be solved for x, y, z. That is,

If in a given region

$$u = f_1(x, y, z), \quad v = f_2(x, y, z), \quad w = f_3(x, y, z)$$

are continuously differentiable functions of x, y, z, the necessary and sufficient condition that these equations be solvable for x, y, z as continuously differentiable functions of u, v, w, is that the Jacobian

$$\nabla u \cdot \nabla v \times \nabla w$$

be different from zero at all points of the region.

21. Divergence and Curl of a Vector. If $\mathbf{F}(x, y, z)$ is a vector function, the scalar

$$\nabla \cdot \mathbf{F} = \mathbf{i} \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{j} \cdot \frac{\partial \mathbf{F}}{\partial y} + \mathbf{k} \cdot \frac{\partial \mathbf{F}}{\partial z} = \operatorname{div} \mathbf{F} \tag{80}$$

is called the **divergence** of $\mathbf{F}$, and the vector

$$\nabla \times \mathbf{F} = \mathbf{i} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{F}}{\partial z} = \operatorname{curl} \mathbf{F} \tag{81}$$

is called the **curl** of $\mathbf{F}$. If

$$\mathbf{F} = \mathbf{i}f_1 + \mathbf{j}f_2 + \mathbf{k}f_3,$$

these have the values

$$\nabla \cdot \mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z},$$

$$\nabla \times \mathbf{F} = \mathbf{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \mathbf{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \mathbf{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right).$$

To obtain a physical picture of these quantities we imagine a fluid flowing in such a way that its velocity $\mathbf{U}(x, y, z)$ at each point $P(x, y, z)$ has precisely the value of the vector $\mathbf{F}$ at that point. We shall first consider two special cases and then show that the general case is a combination of these.

As a first case suppose the motion is simply a rotation with angular velocity

$$\Omega = \mathbf{i}\omega_1 + \mathbf{j}\omega_2 + \mathbf{k}\omega_3$$

about an axis through the origin. In §9 we found that the velocity at $P(x, y, z)$ in such a rotation is

$$\mathbf{U} = \Omega \times \mathbf{R}, \tag{82}$$

where

$$\mathbf{R} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

is the vector from the origin to P. Since Ω is constant, equation (63) gives

$$\nabla \times \mathbf{U} = \Omega (\nabla \cdot \mathbf{R}) - \Omega \cdot \nabla \mathbf{R}.$$

But from equations (68) and (70),

$$\nabla \cdot \mathbf{R} = 3, \quad \Omega \cdot \nabla \mathbf{R} = \Omega.$$

Consequently

$$\nabla \times \mathbf{U} = 2\Omega.$$

Also, from equations (62) and (69),

$$\nabla \cdot \mathbf{U} = \mathbf{R} \cdot \nabla \times \Omega - \Omega \cdot \nabla \times \mathbf{R} = 0.$$

If $\mathbf{U}$ *is the velocity of a rotating fluid, the curl of* $\mathbf{U}$ *is then twice the vector angular velocity of rotation, and the divergence of* $\mathbf{U}$ *is zero.*

Next suppose the fluid undergoes homogeneous stretching or compression parallel to the coördinate axes in such a way that the origin remains fixed, but unit lengths parallel to OX, OY, OZ change at the rates a, b, c respectively. In this motion the point $P(x, y, z)$ has the velocity

$$\mathbf{U} = \mathbf{i}ax + \mathbf{j}by + \mathbf{k}cz \tag{83}$$

and any portion of the fluid changes volume at the rate

$$a + b + c = \nabla \cdot \mathbf{U}$$

per unit volume. Also

$$\nabla \times \mathbf{U} = 0.$$

Under such a homogeneous deformation the velocity therefore has curl equal to zero and divergence equal to the rate of change of volume per unit volume.

Next take the case of arbitrary motion. Consider values of $\mathbf{U}$ in the neighborhood of a point P_0 which for convenience we take as origin of coördinates. If $\mathbf{U}$ and its first derivatives are continuous in this region, by Maclaurin's series the velocity at a neighboring point $P(x, y, z)$ is

$$\mathbf{U} = \mathbf{U}_0 + \mathbf{A}x + \mathbf{B}y + \mathbf{C}z + \cdots, \tag{84}$$

where $\mathbf{U}_0$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are vector constants and the remainder of the series is an infinitesimal of higher than the first order. At P_0

the vector $\mathbf{U}$ then has the same divergence and curl as the quantity

$$\mathbf{W} = \mathbf{A}x + \mathbf{B}y + \mathbf{C}z. \tag{85}$$

Choose axes of coördinates which reduce $\mathbf{W}$ to the simplest form. For that purpose form the product

$$\mathbf{R} \cdot \mathbf{W} = (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) \cdot (\mathbf{A}x + \mathbf{B}y + \mathbf{C}z).$$

This is a homogeneous quadratic in x, y, z with scalar coefficients. For such a quadratic there is always at least one set of principal axes with respect to which it takes the form

$$ax^2 + by^2 + cz^2.$$

Suppose those are the axes of coördinates. Then

$$\mathbf{R} \cdot \mathbf{W} = ax^2 + by^2 + cz^2. \tag{86}$$

Take the gradient of both sides of this equation. By formula (64)

$$\nabla(\mathbf{R} \cdot \mathbf{W}) = \mathbf{R} \cdot \nabla\mathbf{W} + \mathbf{W} \cdot \nabla\mathbf{R} + \mathbf{R} \times (\nabla \times \mathbf{W}) + \mathbf{W} \times (\nabla \times \mathbf{R}).$$

Direct expansion gives

$$\mathbf{R} \cdot \nabla\mathbf{W} = \mathbf{W} \cdot \nabla\mathbf{R} = \mathbf{W}, \qquad \nabla \times \mathbf{R} = 0.$$

Introducing these values, the gradient of (86) becomes

$$2\mathbf{W} + \mathbf{R} \times (\nabla \times \mathbf{W}) = 2\mathbf{i}ax + 2\mathbf{j}by + 2\mathbf{k}cz,$$

whence

$$\mathbf{W} = \tfrac{1}{2}(\nabla \times \mathbf{W}) \times \mathbf{R} + \mathbf{i}ax + \mathbf{j}by + \mathbf{k}cz. \tag{87}$$

The quantity $\nabla \times \mathbf{W}$ is equal to the curl of $\mathbf{U}$ at the point P_0. Denoting this by $(\nabla \times \mathbf{U})_0$, equation (84) is equivalent to

$$\mathbf{U} = \mathbf{U}_0 + \tfrac{1}{2}(\nabla \times \mathbf{U})_0 \times \mathbf{R} + \mathbf{i}ax + \mathbf{j}by + \mathbf{k}cz + \cdots \tag{88}$$

To first order terms the velocity near P_0 is thus the sum of three parts. The first is a common velocity $\mathbf{U}_0$ with which the whole fluid moves without rotation or change of form. The second is a rotation

$$\mathbf{U}' = \tfrac{1}{2}(\nabla \times \mathbf{U})_0 \times \mathbf{R}$$

with angular velocity

$$\Omega = \tfrac{1}{2}(\nabla \times \mathbf{U})_0$$

equal to half the curl at P_0. The third is a velocity

$$\mathbf{U}'' = \mathbf{i}ax + \mathbf{j}by + \mathbf{k}cz$$

which causes unit lengths parallel to OX, OY, OZ to change at the rates a, b, c respectively, and so causes the volume of any portion of the fluid to change at the rate

$$a + b + c$$

per unit volume. By differentiating both sides of (88) it is found that this is the divergence of $\mathbf{U}$ at P_0.

If then we interpret a given vector function $\mathbf{F}(x, y, z)$ as the velocity at $P(x, y, z)$ in a moving fluid, *an infinitesimal portion of the fluid about P will have a vector angular velocity equal to half the curl of $\mathbf{F}$ and a rate of change of volume per unit volume equal to the divergence of $\mathbf{F}$ at P.* This connection with rotation and expansion in a moving fluid is responsible for the terms curl and divergence.

PROBLEMS

1. Let $\mathbf{R} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ be the vector and

$$r = \sqrt{x^2 + y^2 + z^2}$$

the distance from the origin to $P(x, y, z)$. From its geometrical interpretation show that

$$\nabla r = \frac{\mathbf{R}}{r}.$$

2. Show that

$$\nabla(r^n) = nr^{n-2}\mathbf{R}.$$

3. Find a function with gradient equal to $\mathbf{R}$.

4. From the equation

$$\mathbf{R} \cdot \mathbf{R} = r^2$$

by taking differentials show that

$$\mathbf{R} \cdot d\mathbf{R} = r\,dr.$$

5. Show that

$$\nabla = \nabla x \frac{\partial}{\partial x} + \nabla y \frac{\partial}{\partial y} + \nabla z \frac{\partial}{\partial z}.$$

6. At a point where $\phi(x, y, z)$ is continuously differentiable and has a maximum or minimum value show that $\nabla\phi = 0$.

7. If $\nabla\phi = 0$ at all points of a region, show that ϕ is constant in that region.

8. If the density $\rho(x, y, z)$ of a fluid is a function of the pressure $p(x, y, z)$ and

$$\phi(x, y, z) = \int_{p_0}^{p} \frac{dp}{\rho},$$

where p_0 is constant, show that

$$\nabla\phi = \frac{\nabla p}{\rho}.$$

9. If θ is the polar coördinate in the xy-plane, find the direction and magnitude of $\nabla\theta$.

10. If A, B are fixed points and θ is the angle at $P(x, y, z)$ subtended by the line segment AB, show that $\nabla\theta$ is normal to the circle through A, B, P.

11. If r_1, r_2 are the distances from the point $P(x, y)$ on an ellipse to its foci, show that the equation

$$r_1 + r_2 = \text{constant},$$

satisfied by these distances, requires

$$\mathbf{u} \cdot \nabla(r_1 + r_2) = 0,$$

where $\mathbf{u}$ is a unit vector tangent to the curve. By interpreting this equation geometrically show that the tangent makes equal angles with the lines to the foci.

12. If along each direction through $P(x, y, z)$ a line is drawn equal in length to the value of

$$\frac{d\phi}{ds}$$

in that direction, the line being drawn backward when the derivative is negative, show that the ends of these lines lie on a sphere tangent to the surface ϕ = constant at P.

13. Show that the derivative of

$$r = \sqrt{x^2 + y^2 + z^2}$$

is equal to unity in each direction through the origin but that r does not have a gradient satisfying equation (50) at the origin. Why are the results of §17 not valid in this case?

14. In a given plane $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_1'$, $\mathbf{i}_2'$ are two pairs of mutually orthogonal unit vectors and

$$\frac{d\phi}{ds_1}, \quad \frac{d\phi}{ds_2}, \quad \frac{d\phi}{ds_1'}, \quad \frac{d\phi}{ds_2'}$$

are derivatives taken in the directions of those vectors at the same point P. Show that

$$\mathbf{i}_1 \frac{d\phi}{ds_1} + \mathbf{i}_2 \frac{d\phi}{ds_2} = \mathbf{i}_1' \frac{d\phi}{ds_1'} + \mathbf{i}_2' \frac{d\phi}{ds_2'}.$$

15. Find a vector normal to the surface

$$x^2 - xy + yz = 3$$

at the point (1, 2, 2).

16. Let $\mathbf{R}(r, \theta)$ be the vector from the origin to the point P of polar coördinates r, θ in the xy-plane.

(*a*) From the definition of partial derivative show that

$$\mathbf{u} = \frac{\partial \mathbf{R}}{\partial r}, \qquad \mathbf{v} = \frac{1}{r}\frac{\partial \mathbf{R}}{\partial \theta}$$

are unit vectors along $\mathbf{R}$ and perpendicular to $\mathbf{R}$ at P.

(*b*) Show that $\mathbf{u}$ and $\mathbf{v}$ are functions of θ alone and that

$$\frac{d\mathbf{u}}{d\theta} = \mathbf{v}, \qquad \frac{d\mathbf{v}}{d\theta} = -\mathbf{u}.$$

17. A particle moves in the xy-plane. Using the results of the preceding problem express its velocity and acceleration in polar coördinates. From the resulting expressions determine the components of velocity and acceleration along $\mathbf{R}$ and perpendicular to $\mathbf{R}$.

18. Prove the formulas

$$\nabla \cdot \mathbf{R} = 3, \qquad \nabla \times \mathbf{R} = 0, \qquad \mathbf{U} \cdot \nabla \mathbf{R} = \mathbf{U}.$$

19. If $\mathbf{A}$ is a constant vector, show that

$$\nabla(\mathbf{A} \cdot \mathbf{R}) = \mathbf{A}.$$

20. If the components of $\mathbf{A}$ are homogeneous linear functions of x, y, z, show that

$$\mathbf{R} \cdot \nabla \mathbf{A} = \mathbf{A}.$$

21. Prove equation (62).

22. Prove equation (63).

23. If $\mathbf{R}_1$, $\mathbf{R}_2$ are the vectors from the fixed points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ to $P(x, y, z)$, find the values of

$$\nabla \cdot (\mathbf{R}_1 \times \mathbf{R}_2), \qquad \nabla \times (\mathbf{R}_1 \times \mathbf{R}_2), \qquad \nabla(\mathbf{R}_1 \cdot \mathbf{R}_2).$$

24. If $u = f_1(x + y)$, $v = f_2(x + y)$, show that $\nabla u \times \nabla v = 0$.

25. If $u = f_1(x, y)$, $v = f_2(x, y)$, $w = f_3(x, y)$, show that $\nabla u \cdot \nabla v \times \nabla w = 0$.

26. If

$$u = a_1x + a_2y + a_3z,$$
$$v = b_1x + b_2y + b_3z,$$
$$w = c_1x + c_2y + c_3z,$$

the a's, b's, and c's being constant, show that

$$J\left(\frac{u, v, w}{x, y, z}\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

27. Find the divergence and curl of the vector

$$\mathbf{U} = \mathbf{i}(x^2 + yz) + \mathbf{j}(y^2 + zx) + \mathbf{k}(z^2 + xy).$$

28. Find the divergence and curl of

$$\frac{\mathbf{i}x + \mathbf{j}y}{x^2 + y^2}.$$

29. If $\mathbf{R}$ is the vector and r the distance from the origin to $P(x, y, z)$, show that the curl of $f(r)\mathbf{R}$ is zero.

30. For what value of n is the divergence of $r^n\mathbf{R}$ equal to zero?

31. Show that

$$(\mathbf{A} \times \nabla) \cdot \mathbf{R} = 0, \qquad (\mathbf{A} \times \nabla) \times \mathbf{R} = -2\mathbf{A}.$$

CHAPTER III

INTEGRATION

22. Space Curves. In three dimensions a curve usually appears as the set of points with rectangular coördinates satisfying two equations. The curve is called a **regular arc** if the coördinate axes can be so chosen that it is represented by equations of the form

$$y = f(x), \quad z = \phi(x), \quad a \leqq x \leqq b,$$

where for values of x in the interval $a \leqq x \leqq b$, $f(x)$ and $\phi(x)$ are continuous functions of x with continuous first derivatives. The length of such a regular arc is defined by the integral

$$\int \sqrt{dx^2 + dy^2 + dz^2} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx.$$

A **regular curve** is one which consists of a regular arc, or a finite number of regular arcs joined end to end, and which does not cross itself. The regular curve is sufficiently general for applications to physics. We shall therefore ordinarily use the word curve to mean regular curve.

Along a regular curve we can assign a positive direction. Let P_0 be a fixed point on the curve, and $P(x, y, z)$ a variable point. Designate by s the quantity numerically equal to the sum of the lengths of regular arcs in the part P_0P of the curve, and let s be positive or negative according as P_0P extends in the positive or negative direction along the curve. Corresponding to each value of s (within certain limits) there is a definite point. If the curve is not closed, each point P corresponds to a single value of s. But if the curve is closed the same point may be determined by values of s which differ by multiples of the dis-

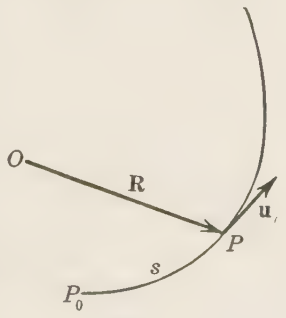


FIG. 27

tance around the curve. In any case the coördinates of P may be considered as functions of s. Likewise any function of x, y, z, whether vector or scalar, is, for points on the curve, a function of s.

Along the curve the vector

$$\mathbf{R} = \overrightarrow{OP}$$

from the origin O to the variable point P, is a continuous function of s. Along each regular arc this function has a continuous derivative

$$\mathbf{u} = \frac{d\mathbf{R}}{ds},$$

which is a vector of unit length extending along the tangent in the direction in which s increases. At a junction of two regular arcs u may however have a discontinuity due to an abrupt change of direction.

The notation

$$d\mathbf{R} = \frac{d\mathbf{R}}{ds}\,ds = \mathbf{u}\,ds$$

is used to represent the vector, tangent at P and having the same length and direction as ds.

23. Line Integral. Let $\mathbf{F}(x, y, z)$ be a vector defined at each point $P(x, y, z)$ of a regular curve AB, s the length of the arc AP, and

$$\mathbf{u} = \frac{d\mathbf{R}}{ds} \tag{89}$$

the unit vector tangent at P. The component of $\mathbf{F}$ along the tangent is

$$F_s = \mathbf{F} \cdot \mathbf{u}, \tag{90}$$

the component being positive when it extends in the same direction as u.

Along the curve this tangential component is a function of s and its integral

$$\int_A^B F_s\,ds = \int_A^B \mathbf{F} \cdot \mathbf{u}\,ds = \int_A^B \mathbf{F} \cdot d\mathbf{R} \tag{91}$$

is called the line integral of $\mathbf{F}$ along the curve from A to B. If AB is a regular arc and $\mathbf{F}$ is continuous along the arc, F_s is a

continuous function of s and so the line integral is an ordinary definite integral of the type treated in calculus. If AB is a regular curve it is a sum of regular arcs, and the line integral is the sum of definite integrals taken in order along those arcs.

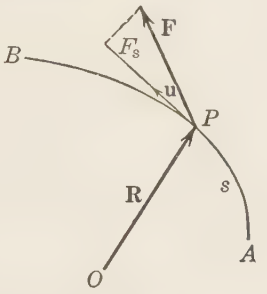


Fig. 28

The element $\mathbf{F} \cdot d\mathbf{R}$ of the line integral just described is a scalar. Other integrals are sometimes used in which the element of integration is a vector. Thus if $\phi(x, y, z)$ is a scalar defined at each point $P(x, y, z)$ of the curve and the unit tangent

$$\mathbf{u} = \mathbf{i}u_1 + \mathbf{j}u_2 + \mathbf{k}u_3$$

has components u_1, u_2, u_3, the notation

$$\int_A^B \phi \, d\mathbf{R} = \int_A^B \phi \frac{d\mathbf{R}}{ds} ds = \int_A^B \phi \mathbf{u} \, ds$$

is used to represent the vector

$$\mathbf{i}\int_A^B \phi u_1 \, ds + \mathbf{j}\int_A^B \phi u_2 \, ds + \mathbf{k}\int_A^B \phi u_3 \, ds,$$

each component of which is the integral of the corresponding component of $\phi\mathbf{u}$. Similarly

$$\int_A^B \mathbf{F} \times d\mathbf{R} = \int_A^B \mathbf{F} \times \mathbf{u} \, ds$$

is used to represent the vector each component of which is obtained by integrating the corresponding component of $\mathbf{F} \times \mathbf{u}$.

24. Case of Line Integral Independent of Path. In general the line integral

$$\int_A^B \mathbf{F} \cdot d\mathbf{R}$$

depends on the curve from A to B along which it is taken. If two different paths join the same end points the integral along one need not be the same as that along the other.

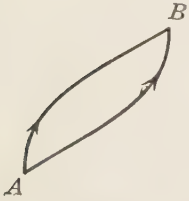


Fig. 29

If, however, $\mathbf{F}$ *is the gradient of a uniform function, the line integral depends only on the end points.* Suppose, in fact,

$$\mathbf{F} = \nabla\phi,$$

where $\phi(x, y, z)$ is one-valued in a certain region. For points in that region

$$\mathbf{F} \cdot d\mathbf{R} = d\mathbf{R} \cdot \nabla\phi = d\phi$$

and

$$\int_A^B \mathbf{F} \cdot d\mathbf{R} = \int_A^B d\phi = \phi(B) - \phi(A),$$

which depends only on the end points.

In that case the integral around a closed path is zero. For such an integral, starting from A and returning to A, can be written

$$\oint \mathbf{F} \cdot d\mathbf{R} = \int_A^A d\phi = \phi(A) - \phi(A) = 0.$$

The notation

$$\oint \mathbf{F} \cdot d\mathbf{R} \tag{92}$$

is used to indicate that the integral is taken around a closed path.

Conversely, *if* $\mathbf{F}(x, y, z)$ *is continuous and*

$$\oint \mathbf{F} \cdot d\mathbf{R} = 0 \tag{93}$$

for every closed path in a three-dimensional region, there is a function $\phi(x, y, z)$ *such that*

$$\mathbf{F} = \nabla\phi. \tag{94}$$

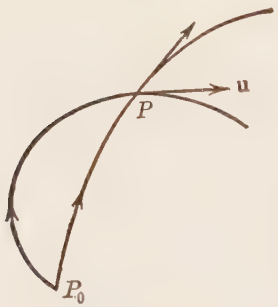


FIG. 30

To show this let

$$\phi(x, y, z) = \int_{P_0}^{P} \mathbf{F} \cdot d\mathbf{R}, \tag{95}$$

where P_0 is a fixed point and $P(x, y, z)$ a variable point in the region. Since the integral around a closed path is zero, ϕ does not depend on the path from P_0 to P. Therefore ϕ is a uniform function of x, y, z. Let P move along a curve through P_0 on which the unit tangent

$$\mathbf{u} = \frac{d\mathbf{R}}{ds}$$

is continuous. Along this curve

$$\phi = \int_{P_0}^{P} \mathbf{F} \cdot d\mathbf{R} = \int_{P_0}^{P} \mathbf{F} \cdot \frac{d\mathbf{R}}{ds} ds$$

is a function of s with derivative

$$\frac{d\phi}{ds} = \mathbf{F} \cdot \frac{d\mathbf{R}}{ds} = \mathbf{F} \cdot \mathbf{u}. \tag{96}$$

The curve P_0P could be chosen so as to have any given direction at P. Hence the derivative of ϕ in any given direction $\mathbf{u}$ is determined by (96). This shows that ϕ has a derivative in each direction at P and that the derivative in any particular direction is a continuous function of x, y, z. Thus ϕ has a definite gradient at P and

$$\frac{d\phi}{ds} = \frac{d\mathbf{R}}{ds} \cdot \nabla\phi = \mathbf{u} \cdot \nabla\phi.$$

Combining this and equation (96) we obtain

$$(\mathbf{F} - \nabla\phi) \cdot \mathbf{u} = 0.$$

Since the last equation holds for every direction of $\mathbf{u}$,

$$\mathbf{F} - \nabla\phi = 0,$$

and so $\mathbf{F}$ is the gradient of ϕ.

25. Work and Potential. A unit mass in a gravitational field is subject to a force $\mathbf{g}$. A unit charge in an electrostatic field is subject to a force $\mathbf{E}$. A unit magnetic pole in a magnetic field is subject to a force $\mathbf{H}$. By the **field intensity**, or force, at P we mean the force which would act on the unit body if it were placed at P. If the force at each point is independent of the time, the field is called static. That is the only case we consider now.

Let $\mathbf{F}(x, y, z)$ be the force at $P(x, y, z)$. The work done by the field when the unit body moves from A to B is

$$\int_A^B \mathbf{F} \cdot d\mathbf{R}.$$

In particular, the work done in moving around a closed path is

$$\oint \mathbf{F} \cdot d\mathbf{R}.$$

If the motion of the unit body around a closed path causes no permanent changes in the field, the law of conservation of energy requires

$$\oint \mathbf{F} \cdot d\mathbf{R} = 0.$$

For if the integral around a particular path is not zero, a machine which causes the unit body to move around that path will receive or lose energy without compensating changes of energy occurring elsewhere.

When the integral around every closed path is zero the force $\mathbf{F}$ is therefore called **conservative.** The work

$$\int_A^B \mathbf{F} \cdot d\mathbf{R}$$

is then independent of the path. If P_0 is a fixed point and $P(x, y, z)$ a variable point, the quantity

$$\phi(x, y, z) = \int_P^{P_0} \mathbf{F} \cdot d\mathbf{R} \tag{97}$$

is called the **potential** at P. It is the work that could be obtained by allowing the unit body starting at P to move to P_0. At P_0 the potential is zero. This position of zero potential can be chosen arbitrarily. In most cases, however, the integral approaches a definite limit

$$\lim_{P_0 \to \infty} \int_P^{P_0} \mathbf{F} \cdot d\mathbf{R}$$

when P_0 moves away indefinitely. In such cases the potential is usually taken to be

$$\phi = \int_P^{\infty} \mathbf{F} \cdot d\mathbf{R}. \tag{98}$$

The potential is a function of the coördinates x, y, z of P. By differentiation we find

$$d\phi = -\mathbf{F} \cdot d\mathbf{R},$$

the negative sign being due to the fact that the variable point P is the lower limit. Also

$$d\phi = d\mathbf{R} \cdot \nabla\phi.$$

Since these equations are true for all values of $d\mathbf{R}$,

$$\mathbf{F} = -\nabla\phi. \tag{99}$$

In a conservative field the force is thus equal to the negative gradient of the potential.

26. Circulation. Let $\mathbf{U}(x, y, z)$ be the velocity at $P(x, y, z)$ in a moving fluid. If it varies with the time this is the velocity at a particular instant of time. The integral

$$\oint \mathbf{U} \cdot d\mathbf{R}$$

around a closed path is called the **circulation** around that path. The average component of velocity along the path is

$$\overline{U}_s = \frac{\oint U_s\, ds}{s} = \frac{\oint \mathbf{U} \cdot d\mathbf{R}}{s},$$

where s is the total length of the path. The circulation is thus

$$\oint \mathbf{U} \cdot d\mathbf{R} = \overline{U}_s s,$$

the product of the length of path and the average velocity along the path. When the circulation is positive the flow is more in the direction of integration than opposite to it. Circulation is thus an indication and to some extent a measure of motion around the path.

Let S be an area on a surface one side of which is considered positive, the other negative. The **positive direction** along the border of S is defined as such that an observer on the positive side of the surface and walking in the positive direction along the

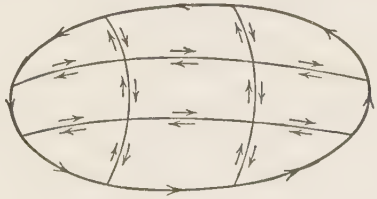


FIG. 31

border always has the area on his left. In a similar way we define the positive direction along the border of any area belonging to S.

Suppose S is cut into smaller areas by a set of arcs, and

$$\Sigma \oint \mathbf{U} \cdot d\mathbf{R}$$

is the sum of integrals in the positive direction around the boundaries of these areas. In this sum the integral along each common boundary of two areas occurs twice with opposite directions, and hence contributes nothing to the result. The integrals which occur only once are those along arcs which belong to the boundary of S. Consequently,

$$\Sigma \oint \mathbf{U} \cdot d\mathbf{R} = \int_C \mathbf{U} \cdot d\mathbf{R}, \tag{100}$$

where C is the boundary of S.

It is to be noted that this equation is valid not only when the boundary C is formed by a single continuous curve but also when it consists of two or more curves as in Fig. 32. In the

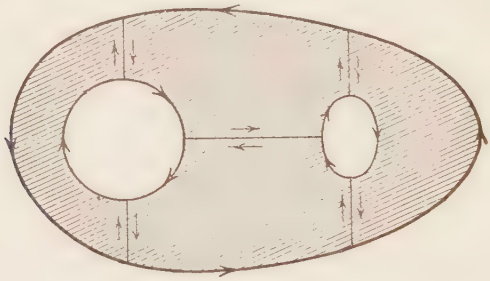


Fig. 32

latter case the right side of (100) is the sum of integrals around the various curves which together constitute the boundary, each integral being taken in the positive direction as defined above.

If then a surface is cut into elementary areas the circulation around the boundary of the surface is equal to the sum of the circulations around the various areas. This is true however small the areas into which the surface is cut. When the elementary areas are infinitesimal we should expect this sum to approach an integral over the surface as limit. In the following sections we shall determine the form of this integral, first discussing integrals in a plane and then extending the results to more general surfaces.

27. Integration in a Plane. At points of a plane region S let

$$\mathbf{F} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$$

be continuous and have continuous first derivatives. Choose coördinates with the plane of S as xy-plane. The positive side of the xy-plane is that on which z (in a right-handed system of coördinates) is positive. The positive direction along the border of an area is then defined as in the preceding section.

We first evaluate the integral

$$\int\int \frac{\partial u}{\partial y}\,dx\,dy$$

over a region (Fig. 33) bounded below and above by regular arcs AB, CD and on the sides by straight lines parallel to the y-axis, the element of integration $dx\,dy$ being considered positive. The value of such an integral can be obtained by integrating with respect to y between the points $P'(x, y')$, $P''(x, y'')$ on the lower and upper curves and then with respect to x between the extreme limits a, b. The first integration gives

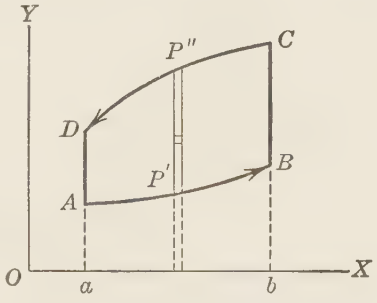


FIG. 33

$$\int\int \frac{\partial u}{\partial y}\,dx\,dy = \int_a^b (u'' - u')dx,$$

where u', u'' are the values of u at P', P'' respectively. Now

$$\int_a^b u'\,dx = \int_A^B u\,dx,$$

the latter integral being taken along the lower curve. Also

$$\int_a^b u''\,dx = \int_D^C u\,dx = -\int_C^D u\,dx,$$

taken along the upper curve. Along BC and DA, x is constant and so

$$\int_B^C u\,dx + \int_D^A u\,dx = 0.$$

Combining the last four equations we get

$$\int\int \frac{\partial u}{\partial y}\,dx\,dy = -\oint u\,dx, \tag{101}$$

the integral on the right being taken in the positive direction around the entire boundary.

If a plane region S can be divided into a finite number of parts each of the kind just discussed, an equation of type (101) is valid for each of them. Adding these equations together we get on the left a double integral over the whole region and on the right the sum of integrals around the boundaries of all the parts. By equation (100) this sum is equal to the integral around the boundary of S. For such a region we then have

$$\int\int \frac{\partial u}{\partial y}\,dx\,dy = -\int_C u\,dx, \tag{102}$$

where C is the boundary of S.

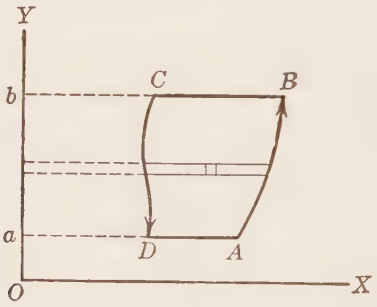


FIG. 34

In a similar way we show that if S can be divided into a finite number of parts of the type $ABCD$ (Fig. 34),

$$\int\int \frac{\partial v}{\partial x}\,dx\,dy = \int_C v\,dy. \tag{103}$$

If the conditions of the preceding discussion are also satisfied, we then have

$$\int\int \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \int_C (u\,dx + v\,dy). \tag{104}$$

For points in the xy-plane,

$$d\mathbf{R} = \mathbf{i}\,dx + \mathbf{j}\,dy.$$

The right side of (104) is therefore the integral

$$\int_C \mathbf{F} \cdot d\mathbf{R}$$

around the boundary of S. The quantity

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

is the z-component of $\nabla \times \mathbf{F}$. Equation (104) is therefore equivalent to

$$\int\int \mathbf{k} \cdot \nabla \times \mathbf{F}\,dx\,dy = \int_C \mathbf{F} \cdot d\mathbf{R}, \tag{105}$$

the integral on the left being taken over an area in the xy-plane and that on the right taken around its boundary. A similar formula can be obtained in case of an area in either of the other coördinate planes.

Before extending these results to a curved surface, it will be necessary to give a brief discussion of integration on such a surface.

28. Surface Integrals. For purposes of integration it is often convenient to divide a surface into simpler portions called **regular surface elements.** By such an element we mean a part of the surface which for some orientation of the coördinate axes has for projection on the xy-plane the interior of a regular closed curve, and which is represented by an equation of the form

$$z = f(x, y),$$

where $f(x, y)$ and its first derivatives with respect to x and y are continuous. An interior point of a regular surface element is called a **regular point** on the surface. We shall consider only surfaces which can be cut into a finite number of regular elements. This is evidently true for the ordinary surfaces, sphere, cylinder, cone, etc. Since an integral over such a surface is merely the sum of integrals over its parts, it will be sufficient at present to discuss integration over a regular surface element S.

At each regular point $P(x, y, z)$ of S there is a definite tangent plane and a normal line with direction cosines proportional to

$$\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}, \quad -1.$$

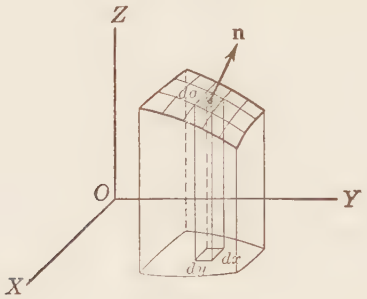


FIG. 35

An element of area da at P may be defined as the area in the tangent plane which has $dx\,dy$ for projection on the xy-plane. That is

$$dx\,dy = \cos\gamma\,da,$$

where γ is the angle between the z-axis and the normal at P. The total area of S is

$$\int_S da = \int\int \sec\gamma\,dx\,dy, \tag{106}$$

the double integral being extended over the projection of S on the xy-plane.

On the surface z may be regarded as a function of x and y. If then $f(x, y, z)$ is a scalar function defined at all points of S, the integral

$$\int_S f\,da$$

may be defined as the double integral

$$\int\int f \sec\gamma\,dx\,dy \tag{107}$$

extended over the projection of S on the xy-plane.

Or the surface integral may be defined directly by a process of summation extended over S. To do this let S be cut into smaller regular elements by means of two systems of curves (for example, the curves in which the planes x = constant, y = constant cut the surface). Let Δa be the area of one of these and f the value of $f(x, y, z)$ at some point Q of this element. Represent by

$$\sum f \, \Delta a \tag{108}$$

the sum for all the elements into which S is cut. If m is the greatest distance between any two points of the same element, we use the notation

$$\Delta a \to 0$$

to express that m approaches zero as limit. If $f(x, y, z)$ is continuous at all points of S it can be shown that when Δa approaches zero the sum approaches a definite limit, independent of the way the surface is cut into regular elements and independent of the choice of points Q in those elements. That limit,

$$\lim_{\Delta a \to 0} \sum f \, \Delta a = \int_S f \, da, \tag{109}$$

which has the same value as (107), is called the integral of f over the surface S.

Let $\mathbf{F}(x, y, z)$ be a vector defined at each point $P(x, y, z)$ of S. Consider one side of the surface positive, the other side negative and let $\mathbf{n}$ be the vector of unit length perpendicular to the surface at P and extending on the positive side. At points of the regular surface element S this unit normal is a continuous vector function of x, y, z. The component of $\mathbf{F}$ along the normal is

$$F_n = \mathbf{F} \cdot \mathbf{n}. \tag{110}$$

If $\mathbf{F}$ is continuous at all points of S this normal component is also continuous. Using F_n instead of f in (109), the resulting integral

$$\lim_{\Delta a \to 0} \sum F_n \, \Delta a = \int_S F_n \, da = \int_S \mathbf{F} \cdot \mathbf{n} \, da \tag{111}$$

is called the **surface integral** of $\mathbf{F}$ over the surface S.

The element $\mathbf{F} \cdot \mathbf{n} \, da$ of the integral just described is a scalar. Other surface integrals are sometimes used in which the element

of integration is a vector. Thus, if $f(x, y, z)$ is a scalar function continuous at all points of S and the unit normal

$$\mathbf{n} = \mathbf{i}n_1 + \mathbf{j}n_2 + \mathbf{k}n_3$$

has components n_1, n_2, n_3, the sum

$$\sum f\mathbf{n}\,\Delta a = \mathbf{i}\sum fn_1\,\Delta a + \mathbf{j}\sum fn_2\,\Delta a + \mathbf{k}\sum fn_3\,\Delta a$$

has components each of which is of the form (109). As Δa approaches zero each of these components approaches a limit and so the sum approaches a limit

$$\lim_{\Delta a \to 0} \sum f\mathbf{n}\,\Delta a = \int_S f\mathbf{n}\,da. \tag{112}$$

Similarly, if $\mathbf{F}(x, y, z)$ is a vector continuous at all points of S,

$$\sum \mathbf{F} \times \mathbf{n}\,\Delta a$$

has a limit

$$\lim_{\Delta a \to 0} \sum \mathbf{F} \times \mathbf{n}\,\Delta a = \int_S \mathbf{F} \times \mathbf{n}\,da, \tag{113}$$

each component of which is the integral of the corresponding component of $\mathbf{F} \times \mathbf{n}$.

To indicate that the integration is taken over a closed surface the notation

$$\oint$$

is sometimes used.

29. Stokes's Theorem. On a surface let C be the curve or finite number of curves forming the complete boundary of an area S. We assume that the surface is two-sided and that S can be resolved into a finite number of regular elements. Choose a positive side of S and let the positive direction along C be that in which an observer on the positive side must move along the boundary if he is to have the area S always on his left. At each regular point $P(x, y, z)$ on the surface let $\mathbf{n}$ be the unit normal drawn toward the positive side. If $\mathbf{F}(x, y, z)$ and its first derivatives with respect to x, y, z are continuous at all points of S and C, we shall show (at least in a very general case) that

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_S \mathbf{n} \cdot \nabla \times \mathbf{F}\, da \tag{114}$$

This is known as Stokes's theorem. It enables us to express an integral around a closed curve as an integral over an area having that curve as boundary.

To prove the theorem we first consider the case of a regular surface element S which has a regular projection on each of the coördinate planes. By this we mean that the projection of S on each coördinate plane is the interior of a regular closed curve, and for points on S each coördinate is a continuous function of the other two with continuous first derivatives. Let

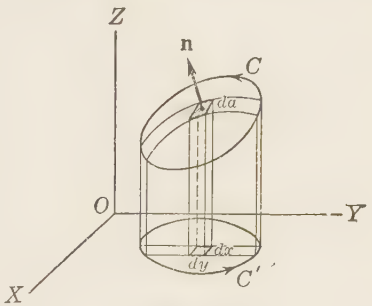


FIG. 36

$$\mathbf{F} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w.$$

Then

$$\int_S \mathbf{n} \cdot \nabla \times \mathbf{F}\, da = \int_S \mathbf{n} \cdot (\nabla \times \mathbf{i}u + \nabla \times \mathbf{j}v + \nabla \times \mathbf{k}w)da.$$

Consider the term

$$\int_S \mathbf{n} \cdot \nabla \times \mathbf{i}u\, da.$$

To evaluate this let da be the surface element which has $dx\, dy$ for projection on the xy-plane. Then

$$\mathbf{n} \cdot \mathbf{k}\, da = dx\, dy.$$

On the surface z is a function of x and y and so

$$u(x, y, z) = \phi(x, y).$$

The partial derivative of this function with respect to y is

$$\frac{\partial \phi}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}.$$

Likewise the vector

$$\mathbf{R} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

is a function of x and y with partial derivative

$$\frac{\partial \mathbf{R}}{\partial y} = \mathbf{j} + \mathbf{k}\frac{\partial z}{\partial y}.$$

This is a vector tangent to the curve cut from the surface by the plane x = constant. It is thus perpendicular to the normal and so

$$\mathbf{n} \cdot \mathbf{j} + \mathbf{n} \cdot \mathbf{k} \frac{\partial z}{\partial y} = 0.$$

By direct expansion we obtain

$$\mathbf{n} \cdot \nabla \times \mathbf{i}u \, \mathrm{da} = \left(\mathbf{n} \cdot \mathbf{j} \frac{\partial u}{\partial z} - \mathbf{n} \cdot \mathbf{k} \frac{\partial u}{\partial y} \right) da.$$

By means of the preceding equations this can be reduced to the form

$$\mathbf{n} \cdot \nabla \times \mathbf{i}u \, da = -\frac{\partial \phi}{\partial y} dx \, dy.$$

Consequently

$$\int_S \mathbf{n} \cdot \nabla \times \mathbf{i}u \, da = -\int\int \frac{\partial \phi}{\partial y} dx \, dy,$$

the integral on the right being taken over the projection of S on the xy-plane. By equation (102) this is equal to the integral

$$\int_{C'} \phi \, dx$$

taken around the curve C' which forms the boundary of the projection. Now ϕ at a point (x, y) of C' has the same value as u at the point (x, y, z) of C. Also in passing between two points of C' the change in x is the same as that in passing between the corresponding points of C. Therefore

$$\int_{C'} \phi \, dx = \int_C u \, dx,$$

and so finally

$$\int_S \mathbf{n} \cdot \nabla \times \mathbf{i}u \, da = \int_C u \, dx.$$

In a similar way we prove

$$\int_S \mathbf{n} \cdot \nabla \times \mathbf{j}v \, da = \int_C v \, dy,$$

$$\int_S \mathbf{n} \cdot \nabla \times \mathbf{k}w \, da = \int_C w \, dz,$$

whence by addition

$$\int_S \mathbf{n} \cdot \nabla \times \mathbf{F}\, da = \int_C \mathbf{F} \cdot d\mathbf{R}.$$

Thus we have proved Stokes's theorem for a surface element which has a regular projection on each coördinate plane. Since the integrals are independent of the coördinate axes the equation remains valid wherever the axes are placed. Now any ordinary surface can be cut into pieces so small that each one for some choice of the axes will have regular projections. Since it is valid for the whole if it is valid for each part, Stokes's theorem follows for any such surface.

30. Vortices. In §21 we found that a fluid of velocity $\mathbf{U}(x, y, z)$ has at $P(x, y, z)$ an instantaneous rotation about an axis parallel to $\nabla \times \mathbf{U}$. A curve which at each of its points is tangent to this instantaneous axis of rotation (that is, tangent to $\nabla \times \mathbf{U}$) is called a vortex line. If through each point of a closed curve we draw the corresponding vortex line, we mark out a tube called a vortex tube. The fluid contained in such a tube is called a vortex. The vortex is considered as having a direction at each point, namely, the direction of $\nabla \times \mathbf{U}$.

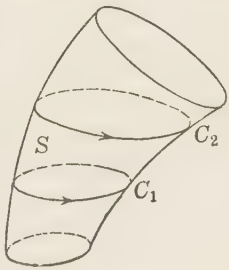


FIG. 37

On the surface of a vortex tube, $\nabla \times \mathbf{U}$ is everywhere perpendicular to the normal. If C_1, C_2 are circuits on the surface and drawn in the same direction once around the tube, C_1 and $-C_2$ constitute the boundary of an area S for which Stokes's theorem gives

$$\int_{C_1} \mathbf{U} \cdot d\mathbf{R} - \int_{C_2} \mathbf{U} \cdot d\mathbf{R} = \int_S \mathbf{n} \cdot \nabla \times \mathbf{U}\, da = 0.$$

The circulation around such a circuit is thus independent of the position of the circuit on the surface of the tube. The circulation is consequently taken as a measure of the strength of the enclosed vortex.

If we divide a cross section A of a vortex into elements of area and through each point of division draw the corresponding vortex line, we resolve the vortex into a sum of vortex filaments.

Applied to such a cross section equation (100), namely,

$$\Sigma \oint \mathbf{U} \cdot d\mathbf{R} = \int_C \mathbf{U} \cdot d\mathbf{R},$$

expresses that the strength of the whole vortex is equal to the sum of the strengths of the filaments of which it is composed.

The equation

$$\int_C \mathbf{U} \cdot d\mathbf{R} = \int_A \mathbf{n} \cdot \nabla \times \mathbf{U}\, da$$

shows that the strength of a vortex is also measured by the integral

$$\int_A \mathbf{n} \cdot \nabla \times \mathbf{U}\, da$$

taken over a cross section. Stokes's theorem thus states that *the circulation around the boundary of any area is equal to the total strength of the vortices crossing that area*, the strength of a vortex being considered positive when it crosses the area in the positive direction of the normal.

31. Flow Across a Surface. Suppose S is a surface of normal $\mathbf{n}$ in a region through which a fluid is moving. The flow across the surface is considered positive at points where the fluid crosses from the negative to the positive side of the surface. We wish to determine the net flow across the surface per unit time.

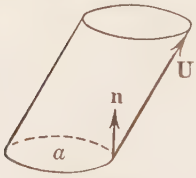


FIG. 38

If the fluid has the same velocity $\mathbf{U}$ at all points, the part which crosses a plane area a (Fig. 38) per unit time forms a cylinder of volume $\mathbf{U} \cdot \mathbf{n}a$, where $\mathbf{n}$ is the unit normal drawn toward the side considered positive. The rate of flow in the positive direction across this area is then

$$\mathbf{U} \cdot \mathbf{n}a = U_n a,$$

where U_n is the component of $\mathbf{U}$ along the normal.

If $\mathbf{U}(x, y, z)$ is continuous at all points of a surface S, divide the surface into elements Δa. The amount which crosses such an element per unit time is

$$\overline{U}_n\, \Delta a,$$

where $\overline{U}_n$ is an average value of U_n at points of Δa. Hence the rate of flow across the surface is

$$\lim_{\Delta a \to 0} \Sigma \overline{U}_n \Delta a = \int_S \mathbf{U} \cdot \mathbf{n}\, da. \tag{115}$$

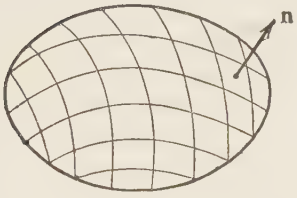


FIG. 39

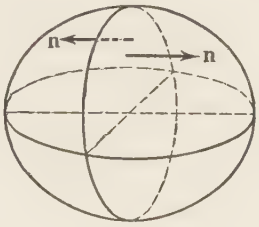


FIG. 40

In integrating over the surface bounding a three-dimensional region it is customary to draw the unit normal outward. Suppose such a region V is cut by surfaces into cells and let

$$\Sigma \int \mathbf{U} \cdot \mathbf{n}\, da$$

be the sum of integrals over the surfaces of all the cells, the normal in each case extending outward from the cell. In this sum a common boundary of two cells will occur twice with normals oppositely directed. These integrals over common boundaries therefore cancel, leaving for the sum the integral over the surface of the whole region. If then S is the surface of V,

$$\int_S \mathbf{U} \cdot \mathbf{n}\, da = \Sigma \int \mathbf{U} \cdot \mathbf{n}\, da. \tag{116}$$

Since this is true however small the parts into which the region is cut, we should expect

$$\int_S \mathbf{U} \cdot \mathbf{n}\, da$$

to be expressible as an integral extended through V. In the following section we shall determine the form of this integral.

32. The Divergence Theorem. If $f(x, y, z)$ is a scalar function defined at each point of a three-dimensional region* V,

* In this book we assume that all regions or sub-regions considered are of the type called regular. For a thorough treatment of these matters the reader is referred to O. D. Kellogg's "Potential Theory."

we use the notation

$$\int_V f\,dv = \int\int\int f\,dx\,dy\,dz$$

to represent the integral of f extended throughout V. The element of volume dv is considered positive.

Let S be the surface or finite number of surfaces forming the complete boundary of V. At each regular point of S let the unit normal n be drawn outward from V. If $\mathbf{F}(x, y, z)$ and its first derivatives with respect to x, y, z are continuous at all points of V and S, we shall show (at least in a very general case) that

$$\int_S \mathbf{F}\cdot\mathbf{n}\,da = \int_V \nabla\cdot\mathbf{F}\,dv. \tag{117}$$

This is known as the divergence theorem. It enables us to express an integral over a closed surface as an integral extended through the volume having that surface as boundary.

Let

$$\mathbf{F} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w.$$

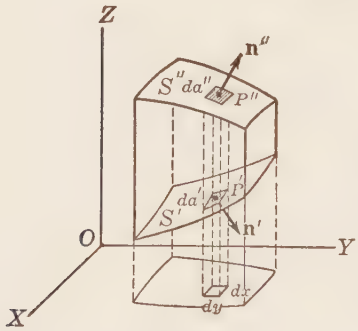


Fig. 41

We first evaluate the integral

$$\int \frac{\partial w}{\partial z}\,dv = \int\int\int \frac{\partial w}{\partial z}\,dx\,dy\,dz$$

for a region V (Fig. 41) bounded below and above by surfaces S', S'' on which z, $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$ are continuous, and on the sides by a cylindrical surface S''' with generators parallel to the z-axis and cutting the xy-plane in a regular closed curve. The value of such an integral can be obtained by integrating with respect to z between the points $P'(x, y, z')$, $P''(x, y, z'')$ on the lower and upper surfaces and then with respect to x, y over the projection of those surfaces on the xy-plane. The first integration gives

$$\int_V \frac{\partial w}{\partial z}\,dv = \int\int (w'' - w')\,dx\,dy, \tag{118}$$

where w', w'' are the values of w at P', P'' respectively. If $\mathbf{n}'$, $\mathbf{n}''$ are the normals drawn outward from V and da', da''

are the surface elements at P', P'' which have $dx\,dy$ for projection on the xy-plane,

$$dx\,dy = da''\,\mathbf{k}\cdot\mathbf{n}'' = -da'\,\mathbf{k}\cdot\mathbf{n}',$$

the negative sign in the last term being due to the downward direction of the normal $\mathbf{n}'$. Hence

$$\int\int w''\,dx\,dy = \int_{S''} \mathbf{k}w\cdot\mathbf{n}\,da,$$

$$\int\int w'\,dx\,dy = -\int_{S'} \mathbf{k}w\cdot\mathbf{n}\,da,$$

and so

$$\int\int (w''-w')dx\,dy = \int_{S'} \mathbf{k}w\cdot\mathbf{n}\,da + \int_{S''} \mathbf{k}w\cdot\mathbf{n}\,da.$$

Also, since $\mathbf{n}$ is perpendicular to $\mathbf{k}$ at all points of the side surface,

$$\int_{S'''} \mathbf{k}w\cdot\mathbf{n}\,da = 0.$$

From (118) and the last two equations we obtain

$$\int_V \frac{\partial w}{\partial z}\,dv = \int_S \mathbf{k}w\cdot\mathbf{n}\,da, \tag{119}$$

where S is the surface forming the complete boundary of V.

If a region can be cut into a finite number of parts each of the type just discussed, equation (119) is valid for each of them. Adding these equations together we get on the left an integral over the whole volume, and on the right a sum of integrals over the surfaces of all the parts. By (116) this sum is equal to the integral over the boundary of the whole region. Equation (119) is thus proved for a region which can be cut into a finite number of parts each of the above type. Taking limits on both sides it also follows for a region which is the limit of a sum of such parts.* It is clear that any ordinary region can be obtained in this way.

* The term limit is here used in the sense that by removing a finite number of the parts the remainder can be made to have a total volume and a total surface as small as we please.

In a similar manner we prove

$$\int_V \frac{\partial u}{\partial x}\,dv = \int_S \mathbf{i}u \cdot \mathbf{n}\,da, \tag{120}$$

$$\int_V \frac{\partial v}{\partial y}\,dv = \int_S \mathbf{j}v \cdot \mathbf{n}\,da. \tag{121}$$

Adding (119), (120), (121), we obtain

$$\int_V \nabla \cdot \mathbf{F}\,dv = \int_S \mathbf{F} \cdot \mathbf{n}\,da,$$

which was to be proved.

If $\mathbf{F}$ is piecewise differentiable in a region V there is an equation of type (117) for each sub-region in which $\mathbf{F}$ is continuously differentiable. If we add these together the surface integrals over the common boundary S_1 of any two sub-regions cancel if $\mathbf{F}$ is continuous on S_1. But if $\mathbf{F}$ has different values on opposite sides of S_1, these integrals do not cancel. For the sum we therefore obtain

$$\int \mathbf{F} \cdot \mathbf{n}\,da = \int_V \nabla \cdot \mathbf{F}\,dv,$$

the integral on the left being taken not only over the boundary of the whole region but also over both sides of each interior surface S_1 on which $\mathbf{F}$ is discontinuous. In the integration over either side of S_1 the normal is directed outward from the sub-region bounded, that is, toward the surface.

33. Sources and Sinks. If $\mathbf{U}(x, y, z)$ is the velocity at $P(x, y, z)$ in a moving fluid and S is the closed surface forming the complete boundary of a region V,

$$\int_S \mathbf{U} \cdot \mathbf{n}\,da$$

represents the net outflow per unit time from that region (cf. §31). If the fluid is incompressible, outflow and inflow are equal and

$$\int_S \mathbf{U} \cdot \mathbf{n}\,da = \int_V \nabla \cdot \mathbf{U}\,dv = 0.$$

Since this is true for every region V,

$$\nabla \cdot \mathbf{U} = 0.$$

The velocity of an incompressible fluid thus has divergence zero.

But if the fluid is compressible, a given portion may change volume as it moves along. The total expansion per unit time within S is then

$$\int_S \mathbf{U} \cdot \mathbf{n}\, da = \int_V \nabla \cdot \mathbf{U}\, dv.$$

The average expansion per unit volume in the region V is

$$\frac{\int_V \nabla \cdot \mathbf{U}\, dv}{\int_V dv}.$$

When the region shrinks to zero about a point P, this approaches the value of $\nabla \cdot \mathbf{U}$ at P. We thus find, as in §21, that *the velocity in a moving fluid has at each point a divergence equal to the rate of change of volume per unit volume at that point.*

In the above discussion we have considered the fluid as nowhere created or destroyed. Fluid could then get into a region only by crossing its boundary. To give it a concrete representation a vector, such as electric or magnetic force which has no connection with fluid motion, is often considered as the velocity of a fluid. If we properly introduce points at which the fluid is emitted or absorbed, any such vector $\mathbf{F}(x, y, z)$ may be considered as the velocity of an incompressible fluid. In a region where $\nabla \cdot \mathbf{F}$ is continuous this is accomplished in the following way. In each element of volume dv where the divergence is positive we imagine sources through which an amount of fluid

$$\nabla \cdot \mathbf{F}\, dv$$

per unit time is admitted to the region. In each element of volume dv where the divergence is negative we imagine sinks through which an amount numerically equal to

$$\nabla \cdot \mathbf{F}\, dv$$

escapes per unit time. The divergence theorem

$$\int_S \mathbf{F} \cdot \mathbf{n}\, da = \int_V \nabla \cdot \mathbf{F}\, dv$$

expresses that *the rate of flow in through the sources and sinks is equal to the rate of flow out through the bounding surface.* By

introducing these sources and sinks the fluid with velocity $\mathbf{F}$ is thus represented as incompressible.

34. Other Integration Formulas. In Stokes's theorem and in the divergence theorem the scalar product of the element of integration and a vector is used. There are analogous formulas involving the cross product of the element of integration with a vector and the product of the element and a scalar. If $f(x, y, z)$ is a scalar and $\mathbf{F}(x, y, z)$ a vector these formulas are:*

$$\int_C f\, d\mathbf{R} = \int_S \mathbf{n} \times \nabla f\, da, \tag{122}$$

$$\int_C d\mathbf{R} \times \mathbf{F} = \int_S (\mathbf{n} \times \nabla) \times \mathbf{F}\, da, \tag{123}$$

$$\int_S f\, \mathbf{n}\, da = \int_V \nabla f\, dv, \tag{124}$$

$$\int_S \mathbf{n} \times \mathbf{F}\, da = \int_V \nabla \times \mathbf{F}\, dv. \tag{125}$$

These, and the earlier formulas involving the dot product, can be symbolically expressed as follows:

$$\int_C d\mathbf{R} \star \Phi = \int_S (\mathbf{n} \times \nabla) \star \Phi\, da, \tag{126}$$

$$\int_S \mathbf{n} \star \Phi\, da = \int_V \nabla \star \Phi\, dv. \tag{127}$$

Here Φ is any quantity, scalar or vector, and $\star$ represents any multiplication (dot, cross, or simple product) which makes sense. The multiplication represented by $\star$ must however be the same on both sides of the equation. In (126) the integral on the right extends over a surface S and that on the left around the complete boundary of S, the directions of $d\mathbf{R}$ and $\mathbf{n}$ being related as in Stokes's theorem. In (127) the integral on the right extends through a region V and that on the left over the complete boundary of V, the normal $\mathbf{n}$ being drawn outward from V. In both cases it is assumed that Φ and its first derivatives with respect to x, y, z are continuous at all points of the region considered.

* In products, such as $(\mathbf{n} \times \nabla) \times \mathbf{F}$, where there are no factors between ∇ and the following parenthesis, we shall understand that the differentiations in ∇ are performed on the factors beyond the parenthesis on the right.

By the use of (126) and (127), we can transform an integral around the boundary of a surface area S into an integral over S, and an integral over the surface of a volume V into an integral extended through V. We can also use the formulas to replace the integral

$$\int_S (\mathbf{n} \times \nabla) \star \Phi \, da \tag{128}$$

by a line integral and to replace

$$\int_V \nabla \star \Phi \, dv \tag{129}$$

by a surface integral, but such a transformation is applicable to a given integral only if it has the form (128) or (129).

The easiest way to prove the formulas is to reduce them to Stokes's theorem and the divergence theorem. Thus to prove (122) let $\mathbf{C}$ be a constant vector. Applying Stokes's theorem to the vector $f\mathbf{C}$ we get

$$\oint f\mathbf{C} \cdot d\mathbf{R} = \int_S \mathbf{n} \cdot \nabla \times f\mathbf{C} \, da = \int_S \mathbf{C} \cdot \mathbf{n} \times \nabla f \, da.$$

Since a constant multiplier can be taken outside the sign of integration we then have

$$\mathbf{C} \cdot \left[\oint f \, d\mathbf{R} - \int_S \mathbf{n} \times \nabla f \, da \right] = 0.$$

Since this is true for all values of $\mathbf{C}$ the expression in brackets must be zero, which was to be proved.

To prove (123) we apply Stokes's theorem to the vector $\mathbf{F} \times \mathbf{C}$. The result is

$$\oint d\mathbf{R} \cdot \mathbf{F} \times \mathbf{C} = \int_S \mathbf{n} \cdot \nabla \times (\mathbf{F} \times \mathbf{C}) da = \int_S \mathbf{n} \cdot [\mathbf{C} \cdot \nabla \mathbf{F} - \mathbf{C} \nabla \cdot \mathbf{F}] \, da.$$

In the first term on the right $\mathbf{n}$ is dotted with $\mathbf{F}$, and in the second term, with $\mathbf{C}$. Hence this is equivalent to

$$\oint \mathbf{C} \cdot d\mathbf{R} \times \mathbf{F} = \int_S \mathbf{C} \cdot (\mathbf{n} \times \nabla) \times \mathbf{F} \, da,$$

whence

$$\mathbf{C} \cdot \left[\oint d\mathbf{R} \times \mathbf{F} - \int_S (\mathbf{n} \times \nabla) \times \mathbf{F} \, da \right] = 0.$$

Since this is true for all values of the constant vector $\mathbf{C}$, we conclude as before that the expression in brackets is zero.

Similarly, by applying the divergence theorem to $f\mathbf{C}$ and $\mathbf{F} \times \mathbf{C}$ we prove (124) and (125).

PROBLEMS

1. Given $\mathbf{F} = \mathbf{j}x - \mathbf{i}y$, find the value of

$$\int \mathbf{F} \cdot d\mathbf{R}$$

along the curve $y^2 = 4x$ from the origin to the point (4, 4).

2. Given $\mathbf{F} = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$, find the value of

$$\oint \mathbf{F} \cdot d\mathbf{R}$$

in the positive direction around the square bounded by the coördinate axes and the lines $x = a$, $y = a$, in the xy-plane.

3. If $\mathbf{W} = \mathbf{i} \sin y + \mathbf{j}x(1 + \cos y)$, find the value of

$$\oint \mathbf{W} \cdot d\mathbf{R}$$

around a circle of radius a in the xy-plane.

4. Find the value of

$$\oint \mathbf{u} \cdot d\mathbf{R}$$

around a circle of radius b if $\mathbf{u}$ is a unit vector tangent at each point.

5. Find the value of

$$\oint (\mathbf{u} \sin s) \cdot d\mathbf{R}$$

around a circle of radius b if s is the arc measured from a fixed point and $\mathbf{u}$ is a unit vector tangent at each point.

6. Show that

$$\oint d\mathbf{R} = 0.$$

7. Show that

$$\oint \mathbf{R} \times d\mathbf{R}$$

taken around a curve in the xy-plane has a numerical value equal to twice the area enclosed by the curve.

8. Show that the function

$$\phi(P) = \int_{P_0}^{P} r^n \mathbf{R} \cdot d\mathbf{R}$$

is independent of the path from P_0 to P and find its value.

9. If $\mathbf{F} = i f(x)$, show that

$$\int_{P_0}^{P} \mathbf{F} \cdot d\mathbf{R}$$

is independent of the path from P_0 to P.

10. Find the vector $\mathbf{F}(x, y, z)$ such that

$$x^2 - y^2 = \int_{(0,0,0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{R}.$$

11. A particle is attracted toward the origin with a force inversely proportional to the square of the distance. Find the potential at $P(x, y, z)$.

12. A particle is attracted toward the origin with a force proportional to the distance. Find a potential.

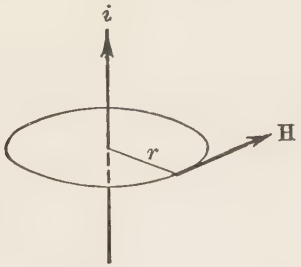
13. A particle of mass m rotates with constant angular velocity about the z-axis. Find a potential associated with centrifugal force.

14. The magnetic force $\mathbf{H}$ due to a current i flowing in a long wire is of magnitude

$$|\mathbf{H}| = \frac{2i}{r}$$

and is directed as shown in the accompanying diagram. Find a potential ϕ such that

$$\mathbf{H} = -\nabla\phi.$$



15. Express the integral

$$\int_C (y\,dx - x\,dy)$$

around a curve C in the xy-plane as a double integral over the area it bounds.

16. Show that

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

is the necessary and sufficient condition that

$$\oint (u\,dx + v\,dy) = 0$$

for every path in the xy-plane.

17. Show directly that

$$\int_S \mathbf{R} \cdot \mathbf{n}\, da = 4\pi r^3,$$

$$\int_S \mathbf{R} \times \mathbf{n}\, da = 0,$$

the integrals being taken over a sphere with center at the origin.

18. Express

$$\int_C (\mathbf{j}x - \mathbf{i}y) \cdot d\mathbf{R}$$

as an integral over an area S having the closed curve C as boundary.

19. If $\mathbf{A}$ is a constant vector, express

$$\int_C \mathbf{A} \times \mathbf{R} \cdot d\mathbf{R}$$

as an integral over a surface having the closed curve C as boundary.

20. Show that

$$\mathbf{k}x = \tfrac{1}{2} \nabla \times (\mathbf{j}x^2).$$

Use this to express

$$\int_S \mathbf{n} \cdot \mathbf{k}x\, da$$

as an integral around the boundary of S.

21. Show that

$$\nabla\phi_1 \times \nabla\phi_2 = \nabla \times (\phi_1 \nabla\phi_2).$$

Use this to express

$$\int_S \mathbf{n} \cdot \nabla\phi_1 \times \nabla\phi_2\, da$$

as an integral around the boundary of S.

22. Given

$$\mathbf{F} = (x^2 + xy - z^2)\mathbf{k},$$

find the value of

$$\int \mathbf{n} \cdot \nabla \times \mathbf{F}\, da$$

over the part of the sphere

$$x^2 + y^2 + z^2 = 1$$

above the xy-plane.

23. Show that

$$\oint \mathbf{n} \cdot \nabla \times \mathbf{F}\, da = 0,$$

the integral being taken over a closed surface on which $\mathbf{F}(x, y, z)$ has continuous first derivatives.

24. If a vector has a constant direction, prove that its curl is perpendicular to that direction.

25. If a vector is normal to a surface at each point, prove that its curl is zero or tangent to the surface.

26. If a vector is zero at each point of a surface, prove that its curl is zero or tangent.

27. The velocity of a fluid is perpendicular to a surface S at all its points. Show that the vortex lines through points of S lie on S.

28. If a, b, c are constants, and

$$\mathbf{F} = a\mathbf{i}x + b\mathbf{j}y + c\mathbf{k}z,$$

find the value of

$$\int \mathbf{F} \cdot \mathbf{n}\, da$$

over a sphere of radius r.

29. Show that

$$\oint \mathbf{R} \cdot \mathbf{n}\, da$$

over any closed surface is equal to 3 times the enclosed volume.

30. Given

$$\mathbf{F} = \mathbf{i}(x^2 - y^2) + 2\mathbf{j}xy + \mathbf{k}(y^2 - xy),$$

find the value of

$$\oint \mathbf{F} \cdot \mathbf{n}\, da$$

over the surface of the cube bounded by $x = 0$, $y = 0$, $z = 0$, $x = 1$, $y = 1$, $z = 1$.

31. Given $\mathbf{F} = \mathbf{i}x - \mathbf{j}y + \mathbf{k}(z^2 - 1)$, find the value of

$$\int \mathbf{F} \cdot \mathbf{n}\, da$$

over the closed surface bounded by the planes $z = 0$, $z = 1$ and the cylinder $x^2 + y^2 = a^2$.

32. Express

$$\oint (x^2 + y^2)(\mathbf{i}x + \mathbf{j}y) \cdot \mathbf{n}\, da$$

over a closed surface in terms of a moment of inertia of the enclosed volume.

33. If a vector $\mathbf{F}$ is directed along the normal at each point of the boundary of a region V, show that

$$\int_V \nabla \times \mathbf{F}\, dv = 0.$$

34. At each point of a closed surface a force of constant magnitude c per unit area acts outward along the normal. Show that the resultant of these forces and their total moment about the origin are both zero.

35. Show that

$$(\mathbf{n} \times \nabla) \times \mathbf{R} = -2\mathbf{n}.$$

Using this relation, express

$$\int \mathbf{n}\, da$$

over any area as an integral around its boundary.

36. Show that

$$\mathbf{n} \times \mathbf{R} = \tfrac{1}{2}\mathbf{n} \times \nabla r^2.$$

Using this relation express

$$\int \mathbf{n} \times \mathbf{R}\, da$$

over any area as an integral around its border.

37. Express

$$\int \mathbf{R}\, dv,$$

extended through a given region, as an integral over its surface.

38. If $\mathbf{C}$ is a constant vector, show that

$$\oint \mathbf{n} \times (\mathbf{C} \times \mathbf{R}) da = 2\mathbf{C}v,$$

where v is the volume enclosed by the surface.

39. If $\mathbf{F}$ is parallel to the xy-plane and C is the boundary of an area in the xy-plane, show that

$$\int_C \mathbf{F} \times d\mathbf{R}$$

is in numerical value equal to the double integral

$$\int\int \nabla \cdot \mathbf{F}\, dx\, dy$$

extended over that area.

40. $\mathbf{U}(x, y, z)$ is the velocity at $P(x, y, z)$ in a fluid of constant density ρ. Show that the part of the fluid which at a particular instant fills a sphere V of radius a and center at the origin has angular momentum about the origin equal to

$$\int_V \tfrac{1}{2}\rho(a^2 - r^2)\nabla \times \mathbf{U}\, dv.$$

41. If $\mathbf{F}$ is continuous and

$$\oint \mathbf{F} \times d\mathbf{R} = 0$$

for every closed curve, show that $\mathbf{F}$ is constant.

42. Show that

$$\int_{S_1}\int_{S_2} (\mathbf{n}_1 \cdot \mathbf{n}_2) r^m \, da_1 \, da_2 = -m(m+1) \int_{V_1}\int_{V_2} r^{m-2} \, dv_1 \, dv_2.$$

In the integral on the right dv_1, dv_2 are the volume elements in the regions V_1, V_2 and r is the distance between them. The integral is the sum for all pairs of such volume elements in the two regions. In the integral on the left $\mathbf{n}_1$, $\mathbf{n}_2$ are unit normals and da_1, da_2 elements of area on the surfaces S_1, S_2 of the two regions and r is the distance between them. It is assumed that m has a value for which the integrals converge.

43. With the same notation as in the preceding problem show that

$$\int_{V_1}\int_{V_2} \frac{dv_1 \, dv_2}{r^2} = -\int_{S_1}\int_{S_2} (\mathbf{n}_1 \cdot \mathbf{n}_2) \ln r \, da_1 \, da_2.$$

44. Show that

$$\int_{C_1}\int_{C_2} r^2 \, d\mathbf{R}_1 \cdot d\mathbf{R}_2 = -4 \int_{S_1} \mathbf{n}_1 \, da_1 \cdot \int_{S_2} \mathbf{n}_2 \, da_2,$$

$$\int_{C_1}\int_{C_2} r^2 \, d\mathbf{R}_1 \times d\mathbf{R}_2 = -2 \int_{S_1} \mathbf{n}_1 \, da_1 \times \int_{S_2} \mathbf{n}_2 \, da_2,$$

the integrals on the right being taken over two surfaces S_1, S_2 and those on the left around their boundaries C_1, C_2.

CHAPTER IV

GENERAL COÖRDINATES

35. Curvilinear Coördinates. In a given region suppose $f_1(x, y, z)$, $f_2(x, y, z)$, $f_3(x, y, z)$ are uniform functions of x, y, z and the equations

$$u_1 = f_1(x, y, z), \quad u_2 = f_2(x, y, z), \quad u_3 = f_3(x, y, z) \tag{130}$$

are solvable for x, y, z as uniform functions

$$x = \phi_1(u_1, u_2, u_3), \quad y = \phi_2(u_1, u_2, u_3), \quad z = \phi_3(u_1, u_2, u_3) \tag{131}$$

of u_1, u_2, u_3 (cf. §20). To each point $P(x, y, z)$ in the region there corresponds a set of values of u_1, u_2, u_3, and to each set of such values (within certain limits) there corresponds a definite point. Such functions u_1, u_2, u_3 are called **curvilinear coördinates.**

In general we assume that u_1, u_2, u_3 are continuously differentiable functions of x, y, z and x, y, z are continuously differentiable functions of u_1, u_2, u_3. In many cases there are however particular points at which these conditions are not satisfied, and it is necessary to exercise care in applying general formulas to regions containing such points.

Through each point P there pass three surfaces

$$u_1 = \text{constant}, \quad u_2 = \text{constant}, \quad u_3 = \text{constant}$$

called coördinate surfaces and these intersect in three curves called coördinate curves. On each coördinate surface one coördinate is constant and two are variable. We designate the surface (as u_1-surface, u_2-surface, u_3-surface) by the coördinate which is constant. On each coördinate curve, one coördinate is variable and two are constant. We designate the curve (as u_1-curve, u_2-curve, u_3-curve) by the one which is variable.

The vector $\mathbf{R}$ from the origin to the variable point $P(x, y, z)$ can be expressed as a function of u_1, u_2, u_3. The partial derivative of this function with respect to u_1 is obtained by varying

P with u_2 and u_3 constant, that is, along the u_1-curve. Therefore

$$\frac{\partial \mathbf{R}}{\partial u_1}$$

is a vector tangent to the u_1-curve at P. Similarly

$$\frac{\partial \mathbf{R}}{\partial u_2}, \quad \frac{\partial \mathbf{R}}{\partial u_3}$$

are tangent to the u_2-curve and u_3-curve respectively.

The partial derivative of $\mathbf{R}$ with respect to u_1 is the ratio of $d\mathbf{R}$ to du_1 when the variations are taken along the u_1-curve. Now in any variation (equation (54))

$$d\mathbf{R} \cdot \nabla u_1 = du_1.$$

Hence

$$\frac{\partial \mathbf{R}}{\partial u_1} \cdot \nabla u_1 = \frac{d\mathbf{R} \cdot \nabla u_1}{du_1} = 1.$$

Similarly,

$$\frac{\partial \mathbf{R}}{\partial u_2} \cdot \nabla u_2 = \frac{\partial \mathbf{R}}{\partial u_3} \cdot \nabla u_3 = 1.$$

In a variation along the u_1-curve, u_2 and u_3 are constant. Consequently ∇u_2 and ∇u_3 are perpendicular to the tangent and so

$$\frac{\partial \mathbf{R}}{\partial u_1} \cdot \nabla u_2 = \frac{\partial \mathbf{R}}{\partial u_1} \cdot \nabla u_3 = 0.$$

In a similar way equations are obtained with u_1, u_2, u_3 interchanged.

These partial derivative relations are all expressed by the equations

$$\frac{\partial \mathbf{R}}{\partial u_i} \cdot \nabla u_i = 1, \quad i = 1, 2, 3 \tag{132}$$

$$\frac{\partial \mathbf{R}}{\partial u_i} \cdot \nabla u_j = 0, \quad i \neq j. \tag{133}$$

These signify that

$$\frac{\partial \mathbf{R}}{\partial u_1}, \quad \frac{\partial \mathbf{R}}{\partial u_2}, \quad \frac{\partial \mathbf{R}}{\partial u_3}$$

and

$$\nabla u_1, \quad \nabla u_2, \quad \nabla u_3$$

are reciprocal sets of vectors. We next consider some of the properties of such reciprocal systems.

36. Reciprocal Systems. $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ and $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$ are called **reciprocal sets**, or reciprocal systems, of vectors if

$$\mathbf{A}_i \cdot \mathbf{B}_i = 1, \quad i = 1, 2, 3 \tag{134}$$

$$\mathbf{A}_i \cdot \mathbf{B}_j = 0, \quad i \neq j. \tag{135}$$

If $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ *and* $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$ *are reciprocal sets of vectors, the triple scalar products* $(\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3)$ *and* $(\mathbf{B}_1\ \mathbf{B}_2\ \mathbf{B}_3)$ *are different from zero.*

For if $(\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3)$ were zero one of the vectors could be expressed as a linear function of the other two. Suppose

$$\mathbf{A}_3 = m_1\mathbf{A}_1 + m_2\mathbf{A}_2.$$

By equation (135) we should then have

$$\mathbf{A}_3 \cdot \mathbf{B}_3 = m_1\mathbf{A}_1 \cdot \mathbf{B}_3 + m_2\mathbf{A}_2 \cdot \mathbf{B}_3 = 0.$$

But this is impossible since equation (134) requires

$$\mathbf{A}_3 \cdot \mathbf{B}_3 = 1.$$

If $(\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3)$ *is not zero the vectors* $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ *have one and only one reciprocal set, namely,*

$$\mathbf{B}_1 = \frac{\mathbf{A}_2 \times \mathbf{A}_3}{(\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3)}, \quad \mathbf{B}_2 = \frac{\mathbf{A}_3 \times \mathbf{A}_1}{(\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3)}, \quad \mathbf{B}_3 = \frac{\mathbf{A}_1 \times \mathbf{A}_2}{(\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3)}. \tag{136}$$

That the vectors determined by (136) form a set reciprocal to $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ may be easily verified by substituting their values in (134) and (135). If there existed a second set $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ reciprocal to the $\mathbf{A}$'s we should have

$$\mathbf{A}_i \cdot \mathbf{C}_i = 1, \quad i = 1, 2, 3$$
$$\mathbf{A}_i \cdot \mathbf{C}_j = 0, \quad i \neq j.$$

Subtracting these equations from (134) and (135) respectively, we obtain

$$\mathbf{A}_i \cdot (\mathbf{B}_j - \mathbf{C}_j) = 0$$

for all values of i and j. If any one of the vectors

$$\mathbf{B}_1 - \mathbf{C}_1, \quad \mathbf{B}_2 - \mathbf{C}_2, \quad \mathbf{B}_3 - \mathbf{C}_3$$

were different from zero, it would therefore be perpendicular to each of the vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$. But this is impossible since $(\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3)$ would then be zero. Consequently the C's and B's are identical and there is only one set reciprocal to the A's.

Expressing the B's in terms of the A's by (136) and using the identity

$$(\mathbf{A}_1 \times \mathbf{A}_2) \cdot (\mathbf{A}_2 \times \mathbf{A}_3) \times (\mathbf{A}_3 \times \mathbf{A}_1) = (\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3)^2,$$

we obtain

$$(\mathbf{B}_1\ \mathbf{B}_2\ \mathbf{B}_3)\ (\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3) = 1. \tag{137}$$

In these formulas either of the reciprocal sets may be replaced by

$$\frac{\partial \mathbf{R}}{\partial u_1}, \quad \frac{\partial \mathbf{R}}{\partial u_2}, \quad \frac{\partial \mathbf{R}}{\partial u_3}$$

and the other by

$$\nabla u_1, \quad \nabla u_2, \quad \nabla u_3.$$

In particular, (137) gives

$$\left(\frac{\partial \mathbf{R}}{\partial u_1}\ \frac{\partial \mathbf{R}}{\partial u_2}\ \frac{\partial \mathbf{R}}{\partial u_3}\right)(\nabla u_1\ \nabla u_2\ \nabla u_3) = 1. \tag{138}$$

By differentiating

$$\mathbf{R} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

with respect to u_1, u_2, u_3 and multiplying the derivatives together we obtain

$$\left(\frac{\partial \mathbf{R}}{\partial u_1}\ \frac{\partial \mathbf{R}}{\partial u_2}\ \frac{\partial \mathbf{R}}{\partial u_3}\right) = \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix} \tag{139}$$

This is the Jacobian of x, y, z with respect to u_1, u_2, u_3 (equation (77)). Also $(\nabla u_1\ \nabla u_2\ \nabla u_3)$ is the Jacobian of u_1, u_2, u_3 with respect to x, y, z. Equation (138) is thus equivalent to

$$J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) J\left(\frac{u_1, u_2, u_3}{x, y, z}\right) = 1. \tag{140}$$

37. Orthogonal Coördinates. If at every point $P(x, y, z)$ the coördinate curves are mutually perpendicular, the coördinates are called **orthogonal.** Because of their simplicity orthogonal coördinates are used almost exclusively.

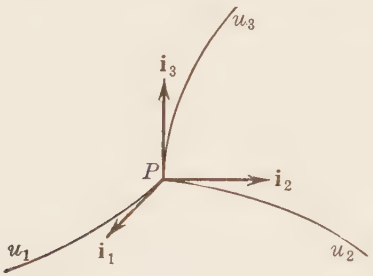


FIG. 42

Suppose then u_1, u_2, u_3 are orthogonal coördinates. Let $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ be unit vectors tangent to the coördinate curves at $P(x, y, z)$ and in each case extending along the curve in the direction in which the corresponding coördinate increases. We assume the subscripts 1, 2, 3 of the coördinates have been so assigned that $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ form a right-handed system of unit vectors. In that case

$$\mathbf{i}_3 = \mathbf{i}_1 \times \mathbf{i}_2. \tag{141}$$

Along each coördinate curve arc length is a function of a single curvilinear coördinate. Let s_1, s_2, s_3 be arc lengths measured along the coördinate curves in the positive directions of u_1, u_2, u_3 respectively and

$$\frac{ds_1}{du_1} = h_1, \quad \frac{ds_2}{du_2} = h_2, \quad \frac{ds_3}{du_3} = h_3 \tag{142}$$

derivatives taken at P. The rectangular parallelepiped with edges

$$ds_1 = h_1\,du_1, \quad ds_2 = h_2\,du_2, \quad ds_3 = h_3\,du_3 \tag{143}$$

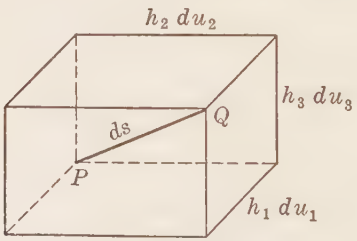


FIG. 43

extending along $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ is called the **element of volume.** It is approximately equal to the infinitesimal region bounded by the

surfaces u_1, u_2, u_3, $u_1 + du_1$, $u_2 + du_2$, $u_3 + du_3$. Its volume is

$$dv = h_1h_2h_3\,du_1\,du_2\,du_3, \tag{144}$$

and its diagonal $ds = PQ$, determined by the equation

$$ds^2 = h_1^2\,du_1^2 + h_2^2\,du_2^2 + h_3^2\,du_3^2, \tag{145}$$

is the element of arc along any curve tangent to PQ at P.

If $\mathbf{R}$ is the vector from the origin to P the partial derivatives of $\mathbf{R}$ with respect to u_1, u_2, u_3 have the directions $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ and have the same magnitudes as the derivatives of arc in (142). Hence

$$\frac{\partial \mathbf{R}}{\partial u_1} = h_1\mathbf{i}_1, \quad \frac{\partial \mathbf{R}}{\partial u_2} = h_2\mathbf{i}_2, \quad \frac{\partial \mathbf{R}}{\partial u_3} = h_3\mathbf{i}_3. \tag{146}$$

These values give

$$\left(\frac{\partial \mathbf{R}}{\partial u_1}\ \frac{\partial \mathbf{R}}{\partial u_2}\ \frac{\partial \mathbf{R}}{\partial u_3}\right) = h_1h_2h_3 \tag{147}$$

as the Jacobian of x, y, z with respect to u_1, u_2, u_3.

Since ∇u_1, ∇u_2, ∇u_3 form a set of vectors reciprocal to the partial derivatives of $\mathbf{R}$ (cf. §35),

$$\nabla u_1 = \frac{\mathbf{i}_1}{h_1}, \quad \nabla u_2 = \frac{\mathbf{i}_2}{h_2}, \quad \nabla u_3 = \frac{\mathbf{i}_3}{h_3}, \tag{148}$$

and these values give

$$(\nabla u_1 \nabla u_2 \nabla u_3) = \frac{1}{h_1h_2h_3} \tag{149}$$

as the Jacobian of u_1, u_2, u_3 with respect to x, y, z.

38. Cylindrical Coördinates. If M is the projection of $P(x, y, z)$ on the xy-plane and r, θ the polar coördinates of M in the xy-plane, the variables $u_1 = r$, $u_2 = \theta$, $u_3 = z$ are called **cylindrical coördinates** of P.

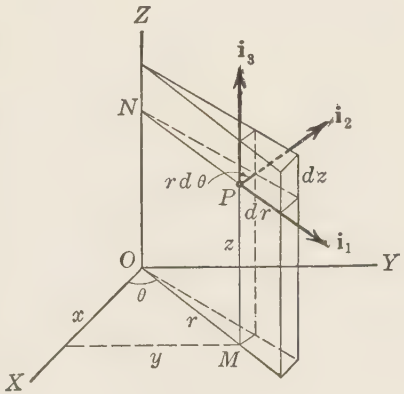


FIG. 44

The relations of cylindrical and rectangular coördinates are shown by a diagram (Fig. 44) or by the equations

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z. \tag{150}$$

At a given point the angle θ may have an infinite number of values differing by multiples of 2π. But in a region which does not extend around or contain points on the z axis, values can be assigned to the cylindrical coördinates so that they are uniform and continuously differentiable functions of x, y, z.

The coördinate surfaces

$$r = \text{constant}, \quad \theta = \text{constant}, \quad z = \text{constant}$$

are cylinders with axis OZ, planes through OZ, and planes perpendicular to OZ. Since these surfaces intersect at right angles the coördinates are orthogonal.

The unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ extend respectively outward along NP, perpendicular to the plane ONP in the direction of increasing θ, and upward. The elements of arc in these directions are

$$ds_1 = dr, \quad ds_2 = r\,d\theta, \quad ds_3 = dz. \tag{151}$$

The quantities h_1, h_2, h_3 in (142) thus have the values

$$h_1 = 1, \quad h_2 = r, \quad h_3 = 1, \tag{152}$$

and the element of volume is

$$dv = r\,dr\,d\theta\,dz. \tag{153}$$

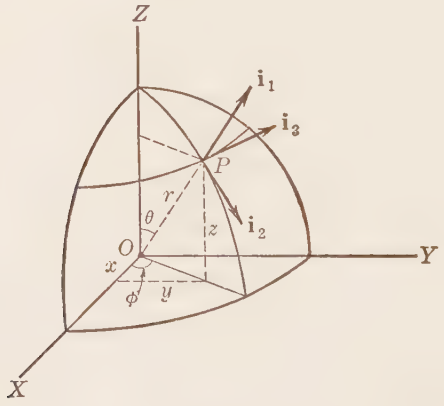


Fig. 45

39. Spherical Coördinates. The spherical, or polar, coördinates of a point P are its distance $r = OP$ from the origin, the angle θ between OP and the z-axis, and the angle ϕ in the

xy-plane between the x-axis and the plane OPZ. Their relations to rectangular coördinates are expressed by the equations

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta. \tag{154}$$

The positive direction of ϕ is that of right-handed rotation about OZ (§5). The positive direction of θ is from OZ toward the terminal side of ϕ. Like all angles these may have many values at the same point. But in a region which does not surround or contain points of the z-axis values can be assigned so that they are uniform and continuously differentiable functions of x, y, z.

The coördinate curves (or lines) through P are the line OP and the two circles cut from the sphere r = constant by the plane ϕ = constant and the cone θ = constant. Since these intersect at right angles the coördinates are orthogonal.

Taking

$$u_1 = r, \quad u_2 = \theta, \quad u_3 = \phi,$$

the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are the normal to the sphere r = constant and the tangents to the two coördinate circles, in each case drawn in the direction in which the corresponding coördinate increases. The elements of arc in these directions are

$$ds_1 = dr, \quad ds_2 = r\,d\theta, \quad ds_3 = r \sin\theta\, d\phi. \tag{155}$$

The quantities h_1, h_2, h_3 thus have the values (equation (142))

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r \sin\theta, \tag{156}$$

and the element of volume is

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi. \tag{157}$$

40. Gradient, Divergence, and Curl. If u_1, u_2, u_3 are curvilinear coördinates, any function of x, y, z is also a function of u_1, u_2, u_3. By equation (59), the gradient of $f(x, y, z)$ may be written

$$\nabla f = \frac{\partial f}{\partial u_1} \nabla u_1 + \frac{\partial f}{\partial u_2} \nabla u_2 + \frac{\partial f}{\partial u_3} \nabla u_3. \tag{158}$$

This can be obtained by expressing f in terms of u_1, u_2, u_3 and applying the operator

$$\nabla u_1 \frac{\partial}{\partial u_1} + \nabla u_2 \frac{\partial}{\partial u_2} + \nabla u_3 \frac{\partial}{\partial u_3} \tag{159}$$

to the resulting function.

In case of a vector

$$\mathbf{F} = \mathbf{i}f_x + \mathbf{j}f_y + \mathbf{k}f_z$$

the components f_x, f_y, f_z are functions of u_1, u_2, u_3. The divergence

$$\nabla \cdot \mathbf{F} = \nabla f_x \cdot \mathbf{i} + \nabla f_y \cdot \mathbf{j} + \nabla f_z \cdot \mathbf{k}$$

and curl

$$\nabla \times \mathbf{F} = \nabla f_x \times \mathbf{i} + \nabla f_y \times \mathbf{j} + \nabla f_z \times \mathbf{k},$$

contain ∇ only in the gradients of the components. In forming these (159) can be used for ∇.

Hence

$$\nabla = \nabla u_1 \frac{\partial}{\partial u_1} + \nabla u_2 \frac{\partial}{\partial u_2} + \nabla u_3 \frac{\partial}{\partial u_3} \tag{160}$$

in the sense that in all the combinations we have defined, the operator ∇ *can be replaced by the right side of* (160).

If the coördinates are orthogonal, ∇u_1, ∇u_2, ∇u_3 have the values given in (148) and

$$\nabla = \frac{\mathbf{i}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\mathbf{i}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\mathbf{i}_3}{h_3}\frac{\partial}{\partial u_3}. \tag{161}$$

The gradient of a scalar function is then

$$\nabla f = \frac{\mathbf{i}_1}{h_1}\frac{\partial f}{\partial u_1} + \frac{\mathbf{i}_2}{h_2}\frac{\partial f}{\partial u_2} + \frac{\mathbf{i}_3}{h_3}\frac{\partial f}{\partial u_3}. \tag{162}$$

A vector $\mathbf{F}$ can be written

$$\mathbf{F} = \mathbf{i}_1 f_1 + \mathbf{i}_2 f_2 + \mathbf{i}_3 f_3, \tag{163}$$

where f_1, f_2, f_3 are its components along the directions $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ respectively. For some purposes it is better to express the vector in a slightly different form. By (141) and (148)

$$\begin{aligned} \mathbf{i}_1 &= \mathbf{i}_2 \times \mathbf{i}_3 = h_2 h_3 \nabla u_2 \times \nabla u_3, \\ \mathbf{i}_2 &= \mathbf{i}_3 \times \mathbf{i}_1 = h_3 h_1 \nabla u_3 \times \nabla u_1, \\ \mathbf{i}_3 &= \mathbf{i}_1 \times \mathbf{i}_2 = h_1 h_2 \nabla u_1 \times \nabla u_2. \end{aligned} \tag{164}$$

Hence

$$\mathbf{F} = (h_2 h_3 f_1)\nabla u_2 \times \nabla u_3 + (h_3 h_1 f_2)\nabla u_3 \times \nabla u_1 + (h_1 h_2 f_3)\nabla u_1 \times \nabla u_2. \tag{165}$$

This seems to be a more complicated expression. Its advantage for certain purposes is due to the fact that each of the products

$\nabla u_2 \times \nabla u_3$, etc. has divergence zero. Thus

$$\nabla \cdot (\nabla u_2 \times \nabla u_3) = \nabla u_3 \cdot (\nabla \times \nabla u_2) - \nabla u_2 \cdot (\nabla \times \nabla u_3) = 0.$$

To find the divergence of $\mathbf{F}$ we use the form (165). As a consequence of the remark just made the divergence of the first term on the right is

$$\nabla(h_2h_3f_1) \cdot \nabla u_2 \times \nabla u_3.$$

Now

$$\nabla(h_2h_3f_1) = \frac{\partial}{\partial u_1}(h_2h_3f_1)\nabla u_1 + \frac{\partial}{\partial u_2}(h_2h_3f_1)\nabla u_2 + \frac{\partial}{\partial u_3}(h_2h_3f_1)\nabla u_3.$$

When the scalar product of this with $\nabla u_2 \times \nabla u_3$ is taken, the last two terms give zero and so

$$\nabla(h_2h_3f_1) \cdot \nabla u_2 \times \nabla u_3 = \frac{\partial}{\partial u_1}(h_2h_3f_1)(\nabla u_1 \nabla u_2 \nabla u_3).$$

Treating the other terms on the right of (165) in a similar way and replacing $(\nabla u_1 \nabla u_2 \nabla u_3)$ by its value from (149) we obtain

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3f_1) + \frac{\partial}{\partial u_2}(h_3h_1f_2) + \frac{\partial}{\partial u_3}(h_1h_2f_3)\right]. \quad (166)$$

To find the curl of $\mathbf{F}$ we express $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ in terms of ∇u_1, ∇u_2, ∇u_3 by (148) and so obtain

$$\mathbf{F} = h_1f_1 \nabla u_1 + h_2f_2 \nabla u_2 + h_3f_3 \nabla u_3. \quad (167)$$

The advantage of this form is that ∇u_1, ∇u_2, ∇u_3 have curl zero and so

$$\nabla \times \mathbf{F} = \nabla(h_1f_1) \times \nabla u_1 + \nabla(h_2f_2) \times \nabla u_2 + \nabla(h_3f_3) \times \nabla u_3. \quad (168)$$

To express this in terms of $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ we substitute

$$\nabla(h_1f_1) = \frac{\partial(h_1f_1)}{\partial u_1}\nabla u_1 + \frac{\partial(h_1f_1)}{\partial u_2}\nabla u_2 + \frac{\partial(h_1f_1)}{\partial u_3}\nabla u_3,$$

whence

$$\nabla(h_1f_1) \times \nabla u_1 = \frac{\partial(h_1f_1)}{\partial u_2}\nabla u_2 \times \nabla u_1 + \frac{\partial(h_1f_1)}{\partial u_3}\nabla u_3 \times \nabla u_1.$$

Replacing the products $\nabla u_2 \times \nabla u_1$, etc. by (164) this becomes

$$\nabla(h_1f_1) \times \nabla u_1 = \frac{\mathbf{i}_2}{h_3h_1}\frac{\partial(h_1f_1)}{\partial u_3} - \frac{\mathbf{i}_3}{h_1h_2}\frac{\partial(h_1f_1)}{\partial u_2}.$$

Expanding the other terms on the right of (168) in a similar way we finally obtain

$$\nabla \times \mathbf{F} = \frac{\mathbf{i}_1}{h_2 h_3}\left(\frac{\partial(h_3 f_3)}{\partial u_2} - \frac{\partial(h_2 f_2)}{\partial u_3}\right) + \frac{\mathbf{i}_2}{h_3 h_1}\left(\frac{\partial(h_1 f_1)}{\partial u_3} - \frac{\partial(h_3 f_3)}{\partial u_1}\right) + \frac{\mathbf{i}_3}{h_1 h_2}\left(\frac{\partial(h_2 f_2)}{\partial u_1} - \frac{\partial(h_1 f_1)}{\partial u_2}\right). \quad (169)$$

41. Volume Integrals. Except for infinitesimals of higher order the elementary region bounded by the coördinate surfaces u_1, u_2, u_3, $u_1 + du_1$, $u_2 + du_2$, $u_3 + du_3$ is a parallelepiped of edges $\frac{\partial \mathbf{R}}{\partial u_1} du_1$, $\frac{\partial \mathbf{R}}{\partial u_2} du_2$, $\frac{\partial \mathbf{R}}{\partial u_3} du_3$ and volume

$$dv = \left(\frac{\partial \mathbf{R}}{\partial u_1}\ \frac{\partial \mathbf{R}}{\partial u_2}\ \frac{\partial \mathbf{R}}{\partial u_3}\right) du_1\, du_2\, du_3. \quad (170)$$

If

$$f(x, y, z) = \phi(u_1, u_2, u_3)$$

is a function of position, the integral

$$\int\int\int f(x, y, z)\, dx\, dy\, dz \quad (171)$$

extended through a particular region is thus equal to

$$\int\int\int \phi(u_1, u_2, u_3)\left(\frac{\partial \mathbf{R}}{\partial u_1}\ \frac{\partial \mathbf{R}}{\partial u_2}\ \frac{\partial \mathbf{R}}{\partial u_3}\right) du_1\, du_2\, du_3 \quad (172)$$

extended through the same region.

The curvilinear coördinates u_1, u_2, u_3 may be considered as rectangular coördinates of a point $P'(u_1, u_2, u_3)$. To each point $P(x, y, z)$ of a region V there corresponds a point P' of a region V'. In this region V' the element of volume is

$$dv' = du_1\, du_2\, du_3.$$

The equality of (171) and (172) expresses that the integral of f over V is equal to the integral of the Jacobian

$$\frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} = \left(\frac{\partial \mathbf{R}}{\partial u_1}\ \frac{\partial \mathbf{R}}{\partial u_2}\ \frac{\partial \mathbf{R}}{\partial u_3}\right)$$

times f, over V'.

In the above discussion we have assumed that the elements of volume dv and dv' are positive. Equation (170) shows that the Jacobian is then positive. We ordinarily restrict the integrals to a region in which the Jacobian does not change sign, and assign the subscripts 1, 2, 3 so that it is positive. When the coördinates are orthogonal, this amounts to choosing the subscripts so that the unit vectors i_1, i_2, i_3 form a right-handed system.

42. Coördinates on a Surface. On a surface S suppose

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v) \tag{173}$$

are uniform and continuously differentiable functions of u, v and that these equations determine u, v as uniform functions of x, y, z. To each point $P(x, y, z)$ on the surface there corresponds a set of values u, v and to each set of such values (within certain limits) there corresponds a definite point on the surface. Such functions u, v are called curvilinear coördinates on the surface.

Through each point on the surface there pass two curves

$$u = \text{constant}, \quad v = \text{constant}$$

called coördinate curves. On each of these one coördinate is constant and one is variable. We designate the curve (as u-curve, v-curve) by the coördinate which is variable.

The vector $\mathbf{R}$ from the origin to the point $P(x, y, z)$ is a function of u and v with differential

$$d\mathbf{R} = \frac{\partial \mathbf{R}}{\partial u} du + \frac{\partial \mathbf{R}}{\partial v} dv. \tag{174}$$

The partial derivatives $\frac{\partial \mathbf{R}}{\partial u}$, $\frac{\partial \mathbf{R}}{\partial v}$ are vectors tangent to the coördinate curves and the product

$$\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}$$

is normal to the surface. In the part of the surface considered we assume that this product is nowhere zero. The vectors $\frac{\partial \mathbf{R}}{\partial u}$, $\frac{\partial \mathbf{R}}{\partial v}$ are then not zero or parallel and so values can be assigned to du and dv such that $d\mathbf{R}$ has any direction in the tangent plane.

The surface gradient of a scalar function $f(u, v)$ is defined as the

vector ∇f in the tangent plane such that

$$d\mathbf{R} \cdot \nabla f = df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv \tag{175}$$

for all values of $d\mathbf{R}$ tangent to the surface at P. To show that there is such a vector let $\mathbf{n}$ be the unit normal at P. By scalar multiplication with the right side of (174) it is seen that

$$\nabla f = \frac{\frac{\partial \mathbf{R}}{\partial v} \times \mathbf{n} \frac{\partial f}{\partial u} + \mathbf{n} \times \frac{\partial \mathbf{R}}{\partial u} \frac{\partial f}{\partial v}}{\left(\frac{\partial \mathbf{R}}{\partial u} \frac{\partial \mathbf{R}}{\partial v} \mathbf{n}\right)} \tag{176}$$

has the required property. That there is only one such vector follows from the fact that (175) determines the component of ∇f along each line in the tangent plane.

By taking for f in (176) the particular values $f = u$, $f = v$, we get

$$\nabla u = \frac{\frac{\partial \mathbf{R}}{\partial v} \times \mathbf{n}}{\left(\frac{\partial \mathbf{R}}{\partial u} \frac{\partial \mathbf{R}}{\partial v} \mathbf{n}\right)}, \quad \nabla v = \frac{\mathbf{n} \times \frac{\partial \mathbf{R}}{\partial u}}{\left(\frac{\partial \mathbf{R}}{\partial u} \frac{\partial \mathbf{R}}{\partial v} \mathbf{n}\right)} \tag{177}$$

Also, since $\mathbf{n}$ is parallel to $\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}$,

$$\mathbf{n} = \frac{\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}}{\left(\frac{\partial \mathbf{R}}{\partial u} \frac{\partial \mathbf{R}}{\partial v} \mathbf{n}\right)}.$$

These equations show that the vectors

$$\nabla u, \quad \nabla v, \quad \mathbf{n}$$

form a set reciprocal to

$$\frac{\partial \mathbf{R}}{\partial u}, \quad \frac{\partial \mathbf{R}}{\partial v}, \quad \mathbf{n}.$$

Equation (176) may be considered as obtained by applying to $f(u, v)$ the operator

$$\nabla = \frac{\frac{\partial \mathbf{R}}{\partial v} \times \mathbf{n} \frac{\partial}{\partial u} + \mathbf{n} \times \frac{\partial \mathbf{R}}{\partial u} \frac{\partial}{\partial v}}{\left(\frac{\partial \mathbf{R}}{\partial u} \frac{\partial \mathbf{R}}{\partial v} \mathbf{n}\right)}. \tag{178}$$

If we regard ∇u and ∇v as determined by the above reciprocal relation, this can be written

$$\nabla = \nabla u \frac{\partial}{\partial u} + \nabla v \frac{\partial}{\partial v}. \tag{179}$$

From (175) it is clear that the surface gradient, and so the surface value of ∇, is independent of coördinates. To obtain a form which is sometimes useful, let du, dv in (178) be the distances ds_1, ds_2 along two perpendicular directions on the surface. The vectors $\frac{\partial \mathbf{R}}{\partial u}$, $\frac{\partial \mathbf{R}}{\partial v}$ are then unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$ which together with $\mathbf{n}$ form an orthogonal system. Hence (178) is equivalent to

$$\nabla = \mathbf{i}_1 \frac{\partial}{\partial s_1} + \mathbf{i}_2 \frac{\partial}{\partial s_2}. \tag{180}$$

A vector $\mathbf{F}$ defined at points of the surface may be regarded as a function of u and v. The **surface divergence** and **surface curl** of $\mathbf{F}$ are obtained by applying to this function the operator (178) or (179).

It is usually possible to define a space function which on the surface has gradient, divergence, or curl equal to the surface gradient, divergence, or curl of a given surface function. For this we assume that the surface has continuous curvature. On each side of the surface there will then be a region such that if P is a point of this region there is a definite shortest line PQ from P to the surface. Let w be the quantity numerically equal to PQ and positive or negative according as P is on the positive or negative side of the surface. We take the u and v of Q and this quantity w as curvilinear coördinates of P. With a scalar f, or vector $\mathbf{F}$, defined at each point $Q(u, v)$ on the surface we associate the space function $f(u, v)$, or $\mathbf{F}(u, v)$, which has at $P(u, v, w)$ the same value as f, or $\mathbf{F}$, at Q. At points of the surface the space gradient of $f(u, v)$ is then equal to the surface gradient of f, and the divergence and curl of $\mathbf{F}(u, v)$ are equal to the surface divergence and surface curl of $\mathbf{F}$.

To show this let $\mathbf{R}$ be the vector from the origin to $P(u, v, w)$. This vector is a function of u, v, w with derivatives

$$\frac{\partial \mathbf{R}}{\partial u}, \quad \frac{\partial \mathbf{R}}{\partial v}, \quad \frac{\partial \mathbf{R}}{\partial w}$$

which form a set of vectors reciprocal to the space gradients

$$\nabla u, \quad \nabla v, \quad \nabla w.$$

At points of the surface

$$\frac{\partial \mathbf{R}}{\partial w} = \nabla w = \mathbf{n}, \tag{181}$$

and so the space gradients of u and v have the same values as the surface gradients (177). When applied to a function of u and v the operator ∇ may be replaced by

$$\nabla u \frac{\partial}{\partial u} + \nabla v \frac{\partial}{\partial v}.$$

Since ∇u and ∇v have the same values whether ∇ represents space or surface differentiation, it follows that all combinations of ∇ with functions of u and v have the same property.

43. Surface Integrals. If u, v are coördinates on a surface the parallelogram with edges $\frac{\partial \mathbf{R}}{\partial u} du$, $\frac{\partial \mathbf{R}}{\partial v} dv$ is called the element of area. Except for infinitesimals of higher order it is equal to the region bounded by the curves u, v, $u + du$, $v + dv$. If it is not zero the vector

$$\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}$$

is normal to the surface. In the part of the surface considered we assume that this vector is nowhere zero and that the coördinates have been so named that it is directed toward the positive side of the surface. If $\mathbf{n}$ is the unit normal and da is the element of area we then have

$$\mathbf{n}\, da = \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} du\, dv. \tag{182}$$

By use of this expression a surface integral can often be expressed as a double integral in the variables u and v.

Consider, for example, the integral

$$\int_S \mathbf{n} \cdot \nabla \times \mathbf{F}\, da$$

taken over a region S of boundary C. From (182) we have

$$\int_S \mathbf{n} \cdot \nabla \times \mathbf{F}\, da = \int\int \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} \cdot \nabla \times \mathbf{F}\, du\, dv.$$

Writing

$$\nabla \times \mathbf{F} = \nabla u \times \frac{\partial \mathbf{F}}{\partial u} + \nabla v \times \frac{\partial \mathbf{F}}{\partial v}$$

and using the equations

$$\frac{\partial \mathbf{R}}{\partial u} \cdot \nabla u = \frac{\partial \mathbf{R}}{\partial v} \cdot \nabla v = 1,$$

$$\frac{\partial \mathbf{R}}{\partial u} \cdot \nabla v = \frac{\partial \mathbf{R}}{\partial v} \cdot \nabla u = 0,$$

satisfied by vectors of reciprocal sets, this becomes

$$\int_S \mathbf{n} \cdot \nabla \times \mathbf{F}\, da = \int\int \left(\frac{\partial \mathbf{F}}{\partial u} \cdot \frac{\partial \mathbf{R}}{\partial v} - \frac{\partial \mathbf{F}}{\partial v} \cdot \frac{\partial \mathbf{R}}{\partial u} \right) du\, dv. \qquad (183)$$

By differentiating and cancelling we obtain

$$\frac{\partial}{\partial u}\left(\mathbf{F} \cdot \frac{\partial \mathbf{R}}{\partial v} \right) - \frac{\partial}{\partial v}\left(\mathbf{F} \cdot \frac{\partial \mathbf{R}}{\partial u} \right) = \frac{\partial \mathbf{F}}{\partial u} \cdot \frac{\partial \mathbf{R}}{\partial v} - \frac{\partial \mathbf{F}}{\partial v} \cdot \frac{\partial \mathbf{R}}{\partial u}.$$

The right side of (183) is thus equal to

$$\int\int \left[\frac{\partial}{\partial u}\left(\mathbf{F} \cdot \frac{\partial \mathbf{R}}{\partial v} \right) - \frac{\partial}{\partial v}\left(\mathbf{F} \cdot \frac{\partial \mathbf{R}}{\partial u} \right) \right] du\, dv.$$

If we consider u and v as rectangular coördinates (x and y) in a plane, by equation (104) this is equal to

$$\int_C \left(\mathbf{F} \cdot \frac{\partial \mathbf{R}}{\partial u}\, du + \mathbf{F} \cdot \frac{\partial \mathbf{R}}{\partial v}\, dv \right).$$

But

$$\frac{\partial \mathbf{R}}{\partial u}\, du + \frac{\partial \mathbf{R}}{\partial v}\, dv = d\mathbf{R}$$

and so (183) is equivalent to

$$\int_S \mathbf{n} \cdot \nabla \times \mathbf{F}\, da = \int_C \mathbf{F} \cdot d\mathbf{R},$$

which is Stokes's theorem.

Next suppose $\mathbf{F}$ is a vector which at points of the region S is tangent to the surface. It might, for example, be the velocity of a fluid flowing along the surface. Since $\mathbf{F}$ is perpendicular to $\mathbf{n}$ we have

$$\mathbf{F} = -\mathbf{n} \times (\mathbf{n} \times \mathbf{F}).$$

Although $\mathbf{F}$ and $\mathbf{n}$ exist only on the surface, we treat them as defined in space (cf. page 93). By equation (181) we then have

$$\nabla \times \mathbf{n} = \nabla \times (\nabla w) = 0. \tag{184}$$

Consequently (equation (62))

$$\nabla \cdot \mathbf{F} = \mathbf{n} \cdot \nabla \times (\mathbf{n} \times \mathbf{F}) \tag{185}$$

and so

$$\int_S \nabla \cdot \mathbf{F}\, da = \int_S \mathbf{n} \cdot \nabla \times (\mathbf{n} \times \mathbf{F})\, da.$$

By Stokes's theorem this is equivalent to

$$\int_S \nabla \cdot \mathbf{F}\, da = \int_C \mathbf{n} \times \mathbf{F} \cdot d\mathbf{R}. \tag{186}$$

This is sometimes called the surface divergence theorem. It is in general true only for vectors $\mathbf{F}$ tangent to the surface. To see its significance consider $\mathbf{F}$ as the velocity of an incompressible fluid flowing over the surface. At a point of the curve C forming the boundary of S

$$d\mathbf{R} \times \mathbf{n} = ds\, \mathbf{n}_1,$$

where ds is the element of arc along C and $\mathbf{n}_1$ is the unit vector tangent to the surface, normal to C, and drawn outward from S. Hence $\mathbf{F} \cdot \mathbf{n}_1$ is the component of velocity perpendicular to $d\mathbf{R}$ and

$$\mathbf{F} \cdot \mathbf{n}_1\, ds = \mathbf{F} \cdot d\mathbf{R} \times \mathbf{n} = \mathbf{n} \times \mathbf{F} \cdot d\mathbf{R}$$

is the rate of flow across $d\mathbf{R}$. The right side of (186) thus represents the rate of flow outward across the curve C and so the left side must represent the rate at which the fluid is supplied from sources inside. The surface divergence $\nabla \cdot \mathbf{F}$ is then the rate of flow per unit area from the surface sources.

PROBLEMS

1. The line from the origin to $P(x, y, z)$ makes angles α, β, γ with the coördinate axes (cf. problem 24, page 27). Show that α, β, γ cannot be used as coördinates of P.

2. Find the set of vectors reciprocal to

$$a\mathbf{i}, \quad b\mathbf{j}, \quad c\mathbf{k},$$

a, b, c being different from zero.

3. Prove that a set of orthogonal unit vectors coincides with its reciprocal set.

4. Find the vectors forming the set reciprocal to

$$\mathbf{A}_1 = \mathbf{i}, \quad \mathbf{A}_2 = \mathbf{i} + \mathbf{j}, \quad \mathbf{A}_3 = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

5. If u_1, u_2, u_3 satisfy a functional relation

$$f(u_1\ u_2, u_3) = 0$$

show that the vectors ∇u_1, ∇u_2, ∇u_3 have no reciprocal set.

6. If each gradient ∇u_1, ∇u_2, ∇u_3 is tangent to the corresponding coördinate curve, prove that u_1, u_2, u_3 are orthogonal coördinates.

7. Show that

$$\frac{\partial}{\partial x}\nabla\phi = \nabla\frac{\partial\phi}{\partial x}.$$

If u_1, u_2, u_3 are curvilinear coördinates, is the equation

$$\frac{\partial}{\partial u_1}\nabla\phi = \nabla\frac{\partial\phi}{\partial u_1}$$

always satisfied?

8. If r, θ, z are cylindrical coördinates, show that

$$\mathbf{i}_1 = \frac{\partial \mathbf{R}}{\partial r}, \quad \mathbf{i}_2 = \frac{1}{r}\frac{\partial \mathbf{R}}{\partial \theta}, \quad \mathbf{i}_3 = \mathbf{k}$$

are the unit vectors tangent to the coördinate curves.

9. Show that the unit vectors in the preceding problem are functions of θ only and

$$\frac{\partial \mathbf{i}_1}{\partial\theta} = \mathbf{i}_2, \qquad \frac{\partial \mathbf{i}_2}{\partial\theta} = -\mathbf{i}_1, \qquad \frac{\partial \mathbf{i}_3}{\partial\theta} = 0.$$

10. If $\mathbf{R}(r, \theta, z)$ is the vector from the origin to a moving point $P(x, y, z)$, use the formulas in the preceding problem to express its acceleration in the form

$$\frac{d^2\mathbf{R}}{dt^2} = a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3.$$

The coefficients a_1, a_2, a_3 are called the components of acceleration in cylindrical coördinates.

11. Verify the equations

$$\nabla\theta + \nabla\times(\mathbf{k}\ln r) = 0,$$
$$\nabla\ln r - \nabla\times(\mathbf{k}\theta) = 0.$$

12. Find the value of the Jacobian

$$J\left(\frac{r, \theta, z}{x, y, z}\right).$$

13. If r, θ, ϕ are spherical coördinates of the point (x, y, z) show that

$$\mathbf{i}_1 = \frac{\partial \mathbf{R}}{\partial r}, \qquad \mathbf{i}_2 = \frac{1}{r}\frac{\partial \mathbf{R}}{\partial \theta}, \qquad \mathbf{i}_3 = \frac{1}{r \sin \theta}\frac{\partial \mathbf{R}}{\partial \phi}$$

are the unit vectors tangent to the coördinate curves.

14. Show that the unit vectors in the preceding problem are functions of θ and ϕ only and

$$\frac{\partial \mathbf{i}_1}{\partial \theta} = \mathbf{i}_2, \qquad \frac{\partial \mathbf{i}_2}{\partial \theta} = -\mathbf{i}_1, \qquad \frac{\partial \mathbf{i}_3}{\partial \theta} = 0,$$

$$\frac{\partial \mathbf{i}_1}{\partial \phi} = \mathbf{i}_3 \sin \theta, \quad \frac{\partial \mathbf{i}_2}{\partial \phi} = \mathbf{i}_3 \cos \theta, \quad \frac{\partial \mathbf{i}_3}{\partial \phi} = -\mathbf{i}_1 \sin \theta - \mathbf{i}_2 \cos \theta.$$

15. If $\mathbf{R}(r, \theta, \phi)$ is the vector from the origin to a moving point $P(r, \theta, \phi)$, use the formulas in the preceding problem to express its acceleration in the form

$$\frac{d^2\mathbf{R}}{dt^2} = a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3.$$

The coefficients a_1, a_2, a_3 are called the components of acceleration in spherical coördinates.

16. Show that

$$\nabla r = \mathbf{i}_1, \qquad \nabla\theta = \frac{\mathbf{i}_2}{r} \qquad \nabla\phi = \frac{\mathbf{i}_3}{r \sin \theta}.$$

17. Verify the equations

$$\nabla\frac{1}{r} = \nabla \times (\cos\theta\, \nabla\phi),$$

$$\nabla\phi = \nabla \times \left(\frac{r\,\nabla\theta}{\sin\theta}\right).$$

18. By computing both Jacobians and taking their product show that

$$J\left(\frac{x, y, z}{r, \theta, \phi}\right) J\left(\frac{r, \theta, \phi}{x, y, z}\right) = 1.$$

19. Express equations (166) and (169) in cylindrical coördinates. Use the resulting equations to determine the divergence and curl of

$$\mathbf{F} = \frac{\mathbf{i}x + \mathbf{j}y}{r^2}.$$

20. Express equations (166) and (169) in spherical coördinates. Use the resulting equations to determine the divergence and curl of

$$\mathbf{F} = \frac{\mathbf{R}}{r^3}.$$

21. On a cylinder with axis OZ show that the surface value of ∇ is

$$\nabla = \frac{\mathbf{i}_2}{r}\frac{\partial}{\partial\theta} + \mathbf{i}_3\frac{\partial}{\partial z}$$

in cylindrical coördinates. If

$$\mathbf{F} = \mathbf{i}_1 f_1 + \mathbf{i}_2 f_2 + \mathbf{i}_3 f_3$$

is a vector defined at all points of the surface, find the surface divergence and curl of $\mathbf{F}$. In particular find the divergence and curl of the unit normal $\mathbf{n}$.

22. On a sphere with center at the origin, show that the surface value of ∇ is

$$\nabla = \frac{\mathbf{i}_2}{r}\frac{\partial}{\partial\theta} + \frac{\mathbf{i}_3}{r\sin\theta}\frac{\partial}{\partial\phi}$$

in spherical coördinates. If

$$\mathbf{F} = \mathbf{i}_1 f_1 + \mathbf{i}_2 f_2 + \mathbf{i}_3 f_3$$

is a vector defined at each point of the sphere, find the surface divergence and curl of $\mathbf{F}$. In particular find the divergence and curl of the unit normal $\mathbf{n}$.

23. On the surface of a cylinder with axis OZ take the rectangular coördinates x, z as curvilinear coördinates. Considering $\mathbf{R}$ as a function $\mathbf{R}(x, z)$ of these coördinates, find the surface values of $\frac{\partial\mathbf{R}}{\partial x}$, $\frac{\partial\mathbf{R}}{\partial z}$, ∇x, and ∇z.

24. On the surface of a sphere with center at the origin take the rectangular coördinates x, y as curvilinear coördinates. Find the surface values of $\frac{\partial\mathbf{R}}{\partial x}$, $\frac{\partial\mathbf{R}}{\partial y}$, ∇x, and ∇y.

CHAPTER V

IRROTATIONAL AND SOLENOIDAL VECTORS

44. Reducible Curves and Surfaces. Most vector quantities in physics belong to two classes called respectively irrotational vectors and solenoidal vectors. In fact we shall show later (§83) that any vector with sufficient continuity can be expressed as the sum of two parts, one of which is irrotational, the other solenoidal.

In discussing these we shall find it convenient to call a closed curve or surface **reducible** in a given region if by continuous deformation it can be shrunk to a point without passing outside the region. In the section of a plane, for example, between two concentric circles a curve surrounding the inner circle is not reducible but all other closed curves are. In the region between two concentric spheres a closed surface is reducible if it does not surround the inner sphere. A region in which all closed curves are reducible is called **simply connected.**

45. Irrotational Vectors. In a three-dimensional region a vector $\mathbf{F}(x, y, z)$ is called **irrotational** if

$$\oint \mathbf{F} \cdot d\mathbf{R} = 0 \tag{187}$$

for every reducible circuit in that region.

If F and its first derivatives are continuous, Stokes's theorem then gives

$$\oint \mathbf{F} \cdot d\mathbf{R} = \int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, da = 0,$$

where S is the area bounded by the circuit. Since this is true for every area,

$$\nabla \times \mathbf{F} = 0. \tag{188}$$

The curl of an irrotational vector is thus zero and conversely, if the curl is zero at all points of a three-dimensional region, the vector is irrotational in that region. If we interpret such a

vector $\mathbf{F}(x, y, z)$ as the velocity at the point $P(x, y, z)$ in a fluid, according to §21 the elementary particles of that fluid have no angular velocity of rotation. That is the reason for the name irrotational.

In a simply connected region where $\mathbf{F}(x, y, z)$ is irrotational and continuous, let P_0 be a fixed point and $P(x, y, z)$ a variable point. For paths from P to P_0 in that region

$$\phi = \int_P^{P_0} \mathbf{F} \cdot d\mathbf{R} \tag{189}$$

is then independent of the path. This integral is therefore a function of x, y, z and, as in §25, we can show that

$$\mathbf{F} = -\nabla\phi. \tag{190}$$

The quantity ϕ is called the **scalar potential** of $\mathbf{F}$. If the given region is not simply connected we can usually find a smaller region about P which is simply connected. For this region (189) and (190) are valid.

A vector irrotational in a given region thus has three fundamental properties:

(1) Its integral around every reducible circuit is zero.
(2) Its curl is zero.
(3) It is the gradient of a scalar function.

For continuously differentiable vectors these properties are equivalent and any one of them might be used as definition of irrotational. For vectors which are only piecewise differentiable, they are however not entirely equivalent. Because it seems most fundamental we use the first of these properties as definition of the irrotational characteristic.

46. Velocity Potential. Let $\mathbf{U}(x, y, z)$ be the velocity at a particular time in a fluid. In a region where there are no vortices the circulation

$$\oint \mathbf{U} \cdot d\mathbf{R}$$

around every reducible circuit is zero (cf. §30). The velocity is therefore an irrotational vector and so the motion is called irrotational.

For such a motion there is a function $\phi(x, y, z)$ which satisfies the equation

$$\mathbf{U} = -\nabla\phi.$$

This function is called the **velocity potential.** It may be determined as the integral

$$\phi = \int_P^{P_0} \mathbf{U} \cdot d\mathbf{R}$$

from the variable point $P(x, y, z)$ to a fixed point P_0, or (since an added constant is usually unimportant) it may be regarded as any definite function with negative gradient equal to $\mathbf{U}$.

When the potential is known the velocity can be obtained by differentiation. If a potential exists it is simpler to describe the motion by means of it rather than the velocity. This is because the potential is a single scalar function whereas the velocity is a set of three scalar functions, namely, its three components.

The velocity is everywhere normal to the surface

$$\phi = \text{constant}.$$

This may give some information about the form of ϕ. For, if the surfaces $\phi_1 = \text{constant}$ and $\phi_2 = \text{constant}$ are the same, ϕ_2 is a function of ϕ_1. Suppose, for example, a fluid rotates about an axis. Its motion is everywhere perpendicular to the planes through the axis. A function which is constant on such a plane is the angle θ between the plane and a fixed plane through the axis. If the motion is irrotational ϕ is then a function of θ. In fact for an incompressible fluid, $\phi = k\theta$, where k is constant.

47. Condition of Integrability. If $\mathbf{F}(x, y, z)$ is not irrotational there may exist a scalar $\mu(x, y, z)$ such that $\mu\mathbf{F}$ is irrotational.

To see the significance of this let

$$\mathbf{F} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$$

and let $\mathbf{R}$ be the vector from the origin to $P(x, y, z)$. Then

$$\mathbf{F} \cdot d\mathbf{R} = u\,dx + v\,dy + w\,dz.$$

If $\mu\mathbf{F}$ is irrotational there is a function $\phi(x, y, z)$ such that

$$\mu\mathbf{F} = \nabla\phi \tag{191}$$

and so

$$\mu\mathbf{F} \cdot d\mathbf{R} = d\mathbf{R} \cdot \nabla\phi = d\phi$$

is an exact differential. Thus μ is an **integrating factor** of $\mathbf{F} \cdot d\mathbf{R}$. Conversely, if μ is such an integrating factor, $\mu\mathbf{F} = \nabla\phi$ is irrotational.

From (191), by taking the curl of both sides, we obtain

$$\mu\nabla \times \mathbf{F} + \nabla\mu \times \mathbf{F} = 0.$$

The scalar product of this with $\mathbf{F}$ is

$$\mu\mathbf{F} \cdot \nabla \times \mathbf{F} = 0.$$

Since μ is not identically zero,

$$\mathbf{F} \cdot \nabla \times \mathbf{F} = 0. \tag{192}$$

If there is an integrating factor, $\mathbf{F}$ must then satisfy this last equation. It can be shown conversely that if this equation is satisfied there is an integrating factor. Equation (192) is therefore called the **condition of integrability.** It is also the condition that there exist a scalar μ such that $\mu\mathbf{F}$ is irrotational.

If there is a family of surfaces

$$\phi(x, y, z) = \text{constant}$$

perpendicular to $\mathbf{F}(x, y, z)$ at every point, $\nabla\phi$ will be parallel to $\mathbf{F}$ and so there will be a scalar μ such that

$$\mu\mathbf{F} = \nabla\phi$$

is irrotational. And conversely if $\mu\mathbf{F}$ is irrotational such surfaces will exist. *The integrability condition is therefore also the condition that there be a family of surfaces ϕ = constant everywhere perpendicular to the vector* $\mathbf{F}$.

48. Solenoidal Vectors. In a given region a vector $\mathbf{F}(x, y, z)$ is called **solenoidal** if

$$\oint \mathbf{F} \cdot \mathbf{n}\, da = 0 \tag{193}$$

for every reducible surface in that region. If $\mathbf{F}$ and its first derivatives are continuous, the divergence theorem then gives

$$\oint \mathbf{F} \cdot \mathbf{n}\, da = \int_V \nabla \cdot \mathbf{F}\, dv = 0.$$

where V is the volume bounded by the closed surface. Since this is true for every volume in the given region

$$\nabla \cdot \mathbf{F} = 0. \tag{194}$$

The divergence of a solenoidal vector is thus zero and conversely, if the divergence is zero at each point of a region, the vector is solenoidal in that region. If the vector is interpreted as the velocity in a fluid, there are then no sources or sinks in the region (cf. §33), and so the stream lines have no beginning or end. The similarity of this to the field about a solenoid is responsible for the term solenoidal.

A vector $\mathbf{F}$ which is the curl of another vector,

$$\mathbf{F} = \nabla \times \mathbf{A}, \tag{195}$$

has divergence zero,

$$\nabla \cdot \mathbf{F} = \nabla \cdot \nabla \times \mathbf{A} = 0,$$

and so is solenoidal. In the following section we shall show conversely that a vector which is solenoidal and continuously differentiable is the curl of another vector.

A vector solenoidal in a given region thus has three fundamental properties:

(1) Its integral over every reducible surface is zero.
(2) Its divergence is zero.
(3) It is the curl of another vector.

For continuously differentiable vectors these properties are equivalent and any one of them might be used as definition of solenoidal. For vectors which are only piecewise differentiable, they are however not entirely equivalent. Because it seems most fundamental we use the first of these properties as definition of the solenoidal characteristic.

49. Vector Potential. We have stated that a vector $\mathbf{F}(x, y, z)$, which in a given region is solenoidal and continuously differentiable, is the curl

$$\mathbf{F} = \nabla \times \mathbf{A} \tag{196}$$

of a vector $\mathbf{A}(x, y, z)$. To prove this we shall show that $\mathbf{F}$ can be expressed as the product of two gradients,

$$\mathbf{F} = \nabla\phi \times \nabla\psi. \tag{197}$$

If then we take

$$\mathbf{A} = \phi \nabla \psi, \tag{198}$$

equation (196) will be satisfied.

To prove (197) let

$$\mathbf{F} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$$

and consider the system of differential equations

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \tag{199}$$

Since u, v, w are continuously differentiable, these equations have two independent solutions

$$u_1(x, y, z) = c_1, \quad u_2(x, y, z) = c_2,$$

where c_1 and c_2 are constants. Either of these solutions represents a surface which at each point is tangent to the vector $\mathbf{F}$. Hence ∇u_1 and ∇u_2 are everywhere perpendicular to $\mathbf{F}$ and so there is a scalar λ such that

$$\mathbf{F} = \lambda \nabla u_1 \times \nabla u_2. \tag{200}$$

Suppose u_3 is a third function which together with u_1 and u_2 forms a system of curvilinear coördinates (§35). Using the form (160) for ∇ we then have

$$\nabla \cdot \mathbf{F} = \frac{\partial \lambda}{\partial u_3} \nabla u_3 \cdot \nabla u_1 \times \nabla u_2.$$

If this is zero,

$$\frac{\partial \lambda}{\partial u_3} = 0,$$

and so λ is a function of u_1 and u_2 only. If

$$\phi = \int \lambda \, du_1,$$

the integration being performed with u_2 and u_3 constant,

$$\nabla \phi = \frac{\partial \phi}{\partial u_1} \nabla u_1 + \frac{\partial \phi}{\partial u_2} \nabla u_2 = \lambda \nabla u_1 + \frac{\partial \phi}{\partial u_2} \nabla u_2.$$

Thus equation (200) is equivalent to

$$\mathbf{F} = \nabla \phi \times \nabla u_2,$$

which has the form (197) required.

If $\mathbf{F}$ is solenoidal there is then a vector $\mathbf{A}$ which satisfies the equation

$$\nabla \times \mathbf{A} = \mathbf{F}. \tag{201}$$

If $\mathbf{F}$ is given this equation does not however uniquely determine $\mathbf{A}$. For if $\mathbf{A}$ is one solution and ϕ is any scalar,

$$\mathbf{A} + \nabla\phi$$

is evidently another solution. If $\mathbf{F}$ does not vary with the time and $\mathbf{A}$ satisfies the equation

$$\nabla \cdot \mathbf{A} = 0 \tag{202}$$

it is called a **vector potential** of $\mathbf{F}$. In a variable field the last equation is however usually replaced by a more complicated condition (cf. equation (410)).

50. Surface Discontinuities. Suppose a region V can be resolved into a finite number of sub-regions in each of which $\mathbf{F}(x, y, z)$ is uniformly continuous. We have called such a vector piecewise continuous in V. Vectors of this type are of frequent occurrence in physics, the electric intensity due to a surface charge being a simple illustration. We assume that the surface of each sub-region consists of a finite number of regular surface elements (§28).

Let V_1, V_2 be two of the sub-regions separated by a surface S and let Q be a regular point on S. Because of its uniform continuity $\mathbf{F}(P)$ approaches a definite limit $\mathbf{F}_1(Q)$ when P, while remaining in V_1, approaches Q as limit. Similarly, $\mathbf{F}(P)$ approaches a limit $\mathbf{F}_2(Q)$ when P approaches Q along a path in V_2. At a point Q where $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal, $\mathbf{F}$ is continuous. If $\mathbf{F}_1(Q)$ and $\mathbf{F}_2(Q)$ have equal components along the normal to S, the normal component of $\mathbf{F}$ is said to be continuous at Q. If $\mathbf{F}_1(Q)$ and $\mathbf{F}_2(Q)$ have equal projections on the tangent plane at Q, the tangential component of $\mathbf{F}$ is said to be continuous at Q.

If $\mathbf{F}$ is irrotational and uniformly continuous in V_1 and in V_2, the necessary and sufficient condition that it be irrotational in $V_1 + V_2$ is that the tangential component of $\mathbf{F}$ be continuous at points of the common boundary S.

To show this we must find the condition that

$$\oint \mathbf{F} \cdot d\mathbf{R} = 0$$

for every reducible circuit in $V_1 + V_2$. By assumption the integral is zero if the circuit lies entirely in V_1 or entirely in V_2. It remains merely to consider circuits which lie partly in V_1 and partly in V_2. Let $ABCD$ (Fig. 46) be such a circuit crossing S at A and C. Join A and C by a path on S and use the symbols $\mathbf{F}_1$, $\mathbf{F}_2$ as above. We then have

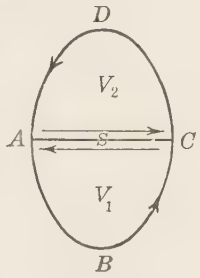


Fig. 46

$$\int_{ABC} \mathbf{F} \cdot d\mathbf{R} + \int_{CA} \mathbf{F}_1 \cdot d\mathbf{R} = 0,$$

$$\int_{CDA} \mathbf{F} \cdot d\mathbf{R} + \int_{AC} \mathbf{F}_2 \cdot d\mathbf{R} = 0,$$

the letters beneath the integral sign in each case indicating the path of integration. Adding and transposing the second terms we have

$$\int_{ABCD} \mathbf{F} \cdot d\mathbf{R} = \int_{CA} (\mathbf{F}_2 - \mathbf{F}_1) \cdot d\mathbf{R}.$$

The integral around the closed path $ABCD$ will then be zero if and only if

$$\int_{CA} (\mathbf{F}_2 - \mathbf{F}_1) \cdot d\mathbf{R} = 0.$$

The condition that this hold for every path CA on S is that $\mathbf{F}_1$ and $\mathbf{F}_2$ have equal tangential components.

In a similar way we show that *if* $\mathbf{F}$ *is solenoidal and uniformly continuous in* V_1 *and in* V_2, *the necessary and sufficient condition that it be solenoidal in* $V_1 + V_2$ *is that the normal component of* $\mathbf{F}$ *be continuous at points of the common boundary.*

From the two theorems we conclude that *if a vector is irrotational, solenoidal, and piecewise continuous in a region, it is continuous in the region.*

PROBLEMS

1. If $\phi(x, y, z)$ and its first derivatives are continuous in a region show that

$$\mathbf{F} = -\nabla\phi$$

is irrotational in the region. If θ is the angle in cylindrical coördinates, why then is

$$\oint \nabla\theta \cdot d\mathbf{R}$$

not zero when the path goes around the z-axis?

2. A fluid moves with constant velocity

$$\mathbf{U} = c\mathbf{i}$$

parallel to the x-axis. Find its velocity potential.

3. A fluid rotates with constant angular velocity ω about the z-axis. Does it have a velocity potential?

4. A fluid flows directly away from the origin, its speed at distance r being

$$cr^n,$$

where c and n are constants. Show that its velocity is irrotational and find the velocity potential.

5. A fluid rotates about the z-axis, its speed at distance r from the axis being

$$\frac{c}{r},$$

where c is constant. Show that its velocity is irrotational and find its velocity potential.

6. If

$$\mathbf{F} = f_1(u_1)\nabla u_1 + f_2(u_2)\nabla u_2,$$

show that $\mathbf{F} \cdot d\mathbf{R}$ has an integrating factor.

7. Determine whether or not the following vectors satisfy the condition of integrability:

(*a*) $\mathbf{i}yz - \mathbf{j}zx + \mathbf{k}(x^2 + y^2)$.

(*b*) $\mathbf{i}y + \mathbf{j}z + \mathbf{k}x$.

(*c*) $\mathbf{i}yz^2 + \mathbf{j}(y^2z - xz^2) - \mathbf{k}y^3$.

(*d*) $\mathbf{i}xy + \mathbf{j}yz - \mathbf{k}zx$.

8. Show that

$$\frac{\mathbf{i}x + \mathbf{j}y}{x^2 + y^2}$$

is a solenoidal vector.

9. If the vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ are both irrotational show that $\mathbf{F}_1 \times \mathbf{F}_2$ is solenoidal.

10. Show that there is no vector $\mathbf{A}$ such that

$$\mathbf{R} = \nabla \times \mathbf{A}.$$

11. In a region V the vector $\mathbf{F}(x, y, z)$ and the scalar $\phi(x, y, z)$ are continuously differentiable. If $\mathbf{F}$ is solenoidal and its normal component is zero on the boundary, show that

$$\int_V \mathbf{F} \cdot \nabla\phi \, dv = 0.$$

12. If r, θ, z are cylindrical coördinates, show that $\nabla\theta$ and $\nabla \ln r$ are solenoidal vectors. Express these vectors in terms of vector potentials (cf. problem 11, page 97).

13. If **R** is the vector from the origin to (x, y, z) and r is its length, show that

$$\nabla \frac{1}{r} = -\frac{\mathbf{R}}{r^3}$$

is solenoidal in any region which does not contain the origin. Find a vector **A** such that (cf. problem 17, page 98)

$$\nabla \frac{1}{r} = \nabla \times \mathbf{A}.$$

Show that a vector **A** which satisfies this condition cannot be uniform and continuously differentiable on a sphere with center at the origin.

14. The curve C forms the complete boundary of an area S. If the integral

$$\int_S \mathbf{F} \cdot \mathbf{n}\, da$$

depends on the curve C but not otherwise on the area S, show that **F** is solenoidal. If **A** is the vector potential of **F**, show that

$$\int_S \mathbf{F} \cdot \mathbf{n}\, da = \int_C \mathbf{A} \cdot d\mathbf{R}.$$

15. If

$$\mathbf{F} = \mathbf{i}f_1 + \mathbf{j}f_2 + \mathbf{k}f_3$$

is a solenoidal vector and

$$\mathbf{A} = \int_{(0, y, 0)}^{(x, y, 0)} \mathbf{j}f_3\, dx + \int_{(x, y, 0)}^{(x, y, z)} (\mathbf{i}f_2 - \mathbf{j}f_1)dz,$$

the path of integration in each case being a straight line, show that

$$\mathbf{F} = \nabla \times \mathbf{A}.$$

16. The vector $\mathbf{F}(z)$ is equal to **k** when z is positive and $-\mathbf{k}$ when z is negative. In a sphere with center at the origin show that **F** is irrotational but not solenoidal.

17. The vector $\mathbf{F}(z)$ is equal to **k** when x is positive and $-\mathbf{k}$ when x is negative. In a sphere with center at the origin show that **F** is solenoidal but not irrotational.

18. In a sphere of radius b and center at the origin a vector has the value

$$\mathbf{F} = \mathbf{R}.$$

Outside the sphere it has the value

$$\mathbf{F} = -b^3 \nabla \frac{1}{r}.$$

Determine whether **F** is irrotational or solenoidal.

19. In the region bounded by the planes $z = \pm h$ and the cylindrical surface

$$x^2 + y^2 = b^2$$

a vector has the value

$$\mathbf{F} = \mathbf{j}x - \mathbf{i}y,$$

and outside this region it is zero. Determine whether $\mathbf{F}$ is irrotational or solenoidal.

20. In a cube with edges parallel to the coördinate axes a vector has the value

$$\mathbf{F} = \mathbf{i}$$

and outside the cube it is zero. In a sphere which contains this cube show that $\mathbf{F}$ is neither irrotational nor solenoidal.

CHAPTER VI

ELECTROSTATIC FIELDS

51. Introduction. In the analysis of vector fields certain quantities occur which have the same mathematical form whatever field is considered. For the presentation of these quantities a single type of field is sufficient. As the one in which the quantities in question seem most completely and naturally represented we take the electrostatic field.

This is the field due to fixed charges distributed in any way in space. If a unit positive charge is placed at the point $P(x, y, z)$ in the field, the fixed charges exert upon it a total force $\mathbf{E}(x, y, z)$ called the **electric intensity** at P. When the unit charge is moved around a closed path and brought back to its starting point no permanent changes are produced in the field. The law of conservation of energy thus requires that the work (§25)

$$\oint \mathbf{E} \cdot d\mathbf{R}$$

done by electric forces in such a motion be zero. $\mathbf{E}$ is therefore an irrotational vector,

$$\nabla \times \mathbf{E} = 0, \tag{203}$$

and the integral

$$\phi = \int_P^{P_0} \mathbf{E} \cdot d\mathbf{R}$$

is independent of the path from P to P_0. If P_0 is kept fixed, ϕ is then a function of the coördinates x, y, z of P which satisfies the equation

$$\mathbf{E} = -\nabla\phi. \tag{204}$$

The quantity ϕ is called the potential at P (cf. §25). It is the work done by the electric intensity when the unit positive charge, on which this force acts, moves from P to P_0. In most

cases this work approaches a limit when P_0 moves away indefinitely. In such cases the potential is usually taken to be

$$\phi = \int_P^\infty \mathbf{E} \cdot d\mathbf{R}. \tag{205}$$

The fundamental problem in the investigation of any irrotational field is the determination of this scalar potential. For when ϕ is known, the vector $\mathbf{E}$ is given immediately by (204). In this chapter we shall consider a number of simple fields and later show that the general case is a combination of these. In this we make extensive use of a geometric entity called the solid angle.

52. Solid Angle. The lines joining a point P to points of an area S generate a figure called a **solid angle.** If a is the area intercepted by these lines on a sphere of center P and radius r, the ratio

$$\omega = \frac{a}{r^2} \tag{206}$$

is taken as the measure of the solid angle. Since similar areas on different spheres are proportional to the squares of the radii, the same value ω is obtained whatever sphere of center P is used. The whole solid angle about a point is equal to the total area of a sphere divided by the square of its radius, or 4π.

Consider one side of S positive, the other negative, and let the unit normal $\mathbf{n}$ be drawn toward the positive side. The solid

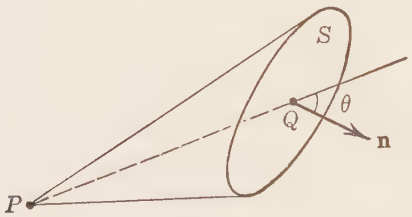


FIG. 47

angle at P subtended by the directed element of area $\mathbf{n}\, da$ at Q is considered positive if the line PQ crosses da from the negative to the positive side, that is, if the angle θ between PQ and the normal at Q is less than 90°.

Let S be a surface which does not pass through P and on which $\mathbf{n}$ is continuous and $\cos\theta$ nowhere zero. We pass from this to the general case by the usual summation and limit processes. If lines are drawn from P to points of the element of area da at Q, these intercept on the sphere of radius PQ, an area approximately equal to the projection

$$\cos\theta\, da$$

on the plane perpendicular to PQ, the error being an infinitesimal of higher order than da. If $\mathbf{R}$ is the vector and r the distance PQ, the differential solid angle is then

$$d\omega = \frac{\cos\theta\, da}{r^2} = \frac{\mathbf{n}\cdot\mathbf{R}\, da}{r^3} \tag{207}$$

and the total solid angle subtended by S is

$$\omega = \int_S \frac{\mathbf{n}\cdot\mathbf{R}\, da}{r^3}. \tag{208}$$

Another way to obtain this same result is to apply the divergence theorem to the region V (Fig. 48) outside a sphere of

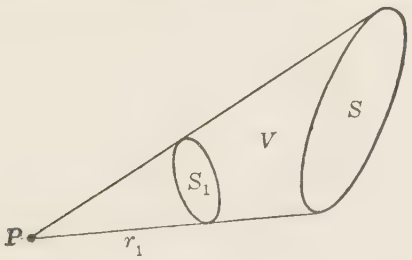


FIG. 48

radius r_1 and inside the cone formed by lines joining P and S. By direct operation it is found that

$$\nabla\cdot\left(\frac{\mathbf{R}}{r^3}\right) = 0. \tag{209}$$

Consequently the integral

$$\int \mathbf{n}\cdot\left(\frac{\mathbf{R}}{r^3}\right) da$$

extended over the whole surface of V is zero. The integral over the conical side surface is zero since there n is perpendicular to **R**. The integrals over S and S_1 are therefore equal and so

$$\int_S \frac{\mathbf{n} \cdot \mathbf{R}\, da}{r^3} = \int_{S_1} \frac{\mathbf{n} \cdot \mathbf{R}\, da}{r^3} = \frac{a_1}{r_1^2} = \omega,$$

where a_1 is the area of S_1.

If S forms the complete boundary of a three-dimensional region, the total solid angle subtended by S at P is zero if P lies outside the region, and 4π if P lies inside.

One way to see this is to consider space cut into elementary cones with vertices at P. If P lies outside S, each cone will cut S an even number of times. The surface elements S_1, S_2, etc., cut out by such a cone, subtend at P solid angles which are numerically equal but alternately positive and negative. Their sum is consequently zero. A similar statement applies to each elementary cone. Hence the total solid angle subtended by S at an outside point is zero.

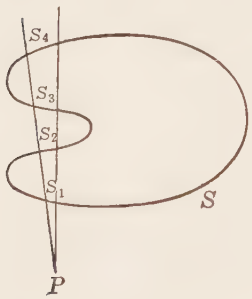


FIG. 49

If however P lies inside S, each elementary cone will cut the surface an odd number of times, the solid angles at the first and last intersections being always positive. Hence the parts of S inside such a cone subtend at P solid angles which have for sum that of the cone. The total solid angle subtended at an inside point is thus the sum of the angles of the elementary cones through the point, namely 4π.

Another way to obtain these results is to apply the divergence theorem to the integral

$$\int_S \frac{\mathbf{n} \cdot \mathbf{R}\, da}{r^3}.$$

If P lies outside S, the function

$$\frac{\mathbf{R}}{r^3}$$

and its first derivatives with respect to x, y, z are continuous

at all points of the region V bounded by S and so (equations (208), (209))

$$\omega = \int_S \frac{\mathbf{n} \cdot \mathbf{R}\, da}{r^3} = \int_V \nabla \cdot \left(\frac{\mathbf{R}}{r^3}\right) dv = 0.$$

But if P lies inside V, we construct a small sphere of center P and radius r_1 less than the shortest distance from P to S. In the region V_1 between S_1 and S, the function

$$\frac{\mathbf{R}}{r^3}$$

and its first derivatives are continuous and so

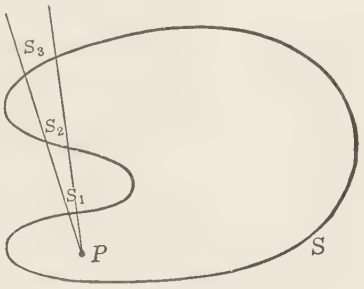


FIG. 50

$$\int_S \frac{\mathbf{n} \cdot \mathbf{R}\, da}{r^3} + \int_{S_1} \frac{\mathbf{n} \cdot \mathbf{R}\, da}{r^3} = \int_{V_1} \nabla \cdot \left(\frac{\mathbf{R}}{r^3}\right) dv = 0.$$

At points of S_1 the normal is directed toward P. Consequently

$$\int_{S_1} \frac{\mathbf{n} \cdot \mathbf{R}\, da}{r^3} = -\frac{4\pi r_1^2}{r_1^2} = -4\pi,$$

whence

$$\int_S \frac{\mathbf{n} \cdot \mathbf{R}\, da}{r^3} = 4\pi$$

which was to be shown.

The quantity r in the integral

$$\omega = \int_S \frac{\mathbf{n} \cdot \mathbf{R}\, da}{r^3}$$

is the distance between the point P at which the solid angle is measured and the point Q at which the element of integration da is located. Sometimes one of these points will be variable and sometimes the other. To have a notation which shows at once to which point a coördinate refers, we shall use x, y, z for the coördinates of P and ξ, η, ζ for those of Q. Then

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}. \tag{210}$$

A function $f(x, y, z, \xi, \eta, \zeta)$ containing all six coördinates may be differentiated with respect to x, y, z (with ξ, η, ζ constant) or with respect to ξ, η, ζ (with x, y, z constant). In such cases we use the notation

$$\nabla_P = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{211}$$

to indicate that the differentiation is with respect to the coördinates of P, and

$$\nabla_Q = \mathbf{i}\frac{\partial}{\partial \xi} + \mathbf{j}\frac{\partial}{\partial \eta} + \mathbf{k}\frac{\partial}{\partial \zeta} \tag{212}$$

to indicate that it is with respect to those of Q.

By direct operation we find

$$\nabla_P \frac{1}{r} = \frac{\mathbf{R}}{r^3}, \tag{213}$$

$$\nabla_Q \frac{1}{r} = -\frac{\mathbf{R}}{r^3}, \tag{214}$$

where

$$\mathbf{R} = \mathbf{i}(\xi - x) + \mathbf{j}(\eta - y) + \mathbf{k}(\zeta - z) \tag{215}$$

is the vector $\overrightarrow{PQ}$. Also, if $f(r)$ is any function of r,

$$\nabla_P f(r) = -\nabla_Q f(r).$$

The solid angle subtended at P can then be written

$$\omega = \int_S \mathbf{n} \cdot \nabla_P \frac{1}{r}\, da = -\int_S \mathbf{n} \cdot \nabla_Q \frac{1}{r}\, da. \tag{216}$$

In the process of integration the variables are the coördinates of Q. *When the symbol ∇ appears under an integral sign with no subscript it therefore indicates differentiation with respect to the coördinates of the point at which the element of integration is located.* Thus we usually write

$$\omega = -\int_S \mathbf{n} \cdot \nabla \frac{1}{r}\, da \tag{217}$$

as the expression for solid angle.

At a point $P(x, y, z)$ which does not belong to S the integrand in (217) is a continuous function of x, y, z and so $\omega(x, y, z)$ is also

continuous. When P approaches a regular point Q of the surface from the positive side (that is, the side toward which the unit normal n is drawn) ω approaches a limit $\omega(Q_+)$, and when P approaches Q from the negative side ω approaches a limit $\omega(Q_-)$. These limits satisfy the equation

$$\omega(Q_+) - \omega(Q_-) = -4\pi. \tag{218}$$

If S is closed this is evident since ω has the constant value zero on the positive side and 4π on the negative side. If S is not closed it can be closed by the addition of a surface S' which does not pass through Q. If $\omega'(x, y, z)$ is the solid angle subtended by S' at $P(x, y, z)$, the sum $\omega + \omega'$ then has the value zero on the positive side and 4π on the negative side. Since ω' is continuous at Q, ω then has definite limits at Q and these limits satisfy (218).

In a region V which lies on one side of S, but may have regular points of S on its boundary, ω has continuous derivatives of all orders. If S is closed this is true because ω is constant in V. If S is not closed it may be closed as above by the addition of a surface S' which has no point in common with V. Since ω' and $\omega + \omega'$ have continuous derivatives at all points of V the same is then true of ω.

Let $f(x, y, z)$ be a function continuous in a region V. Cut the region into cones with a common vertex O and these into cells by spheres of center O. The cell bounded by a cone of solid angle $d\omega$ and two spheres of radii r, $r + dr$ is approximately a cylinder of base $r^2\, d\omega$, altitude dr, and volume $r^2\, d\omega\, dr$. Multiply the volume Δv of each cell by the value of f at some one of its points and form the sum

$$\Sigma f\, \Delta v$$

for all the cells. If the process of division is continued in any way so that the cells approach zero in all their dimensions, this sum approaches a limit

$$\int_V f\, dv = \int\int_V f r^2\, d\omega\, dr \tag{219}$$

independent of the precise way the elementary cones of vertex O are chosen.

For general discussions this form of expression is convenient.

But if it is necessary to find the actual value of such an integral (which is unusual) some definite method of division must be adopted. The common procedure is to use spherical coördinates with O as origin. In these coördinates the element of volume is (cf. §39)

$$dv = r^2 \sin\theta \, d\theta \, d\phi \, dr,$$

the solid angle which it subtends at the origin is

$$d\omega = \sin\theta \, d\theta \, d\phi,$$

and the integral of a function $f(r, \theta, \phi)$ over a region V is

$$\int\int_V fr^2 \, d\omega \, dr = \int\int\int fr^2 \sin\theta \, d\theta \, d\phi \, dr,$$

the limits in the triple integral being chosen so that it extends throughout V.

53. Point Charges. As the simplest type of field we begin with that about a point charge. By Coulomb's law two charges e_1, e_2 of the same sign repel (two of opposite sign attract) each other with a force along the joining line numerically equal to

$$\frac{e_1 e_2}{r^2}, \tag{220}$$

where r is the distance between them.

Let the charge e be located at the point $Q(\xi, \eta, \zeta)$ and let

$$\mathbf{R} = \mathbf{i}(x - \xi) + \mathbf{j}(y - \eta) + \mathbf{k}(z - \zeta) \tag{221}$$

be the vector and

$$r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2} \tag{222}$$

the distance from Q to $P(x, y, z)$. By Coulomb's law the force exerted upon a unit charge at P, that is, the electric intensity at P due to this charge at Q, is

$$\mathbf{E} = \frac{e\mathbf{R}}{r^3} = -e\nabla_P \frac{1}{r}. \tag{223}$$

It is to be noted that $\mathbf{R}$ is here the negative of the vector used in §52. This change of direction is due to the fact that the solid angle is seen from P whereas the electric intensity emanates from Q.

Consider the charge e and point Q fixed. The vector $\mathbf{E}$ is then a function of the coördinates x, y, z of P. Using ∇ to express differentiation with respect to the coördinates of P, by direct operation we find

$$\nabla \cdot \mathbf{E} = \nabla \times \mathbf{E} = 0.$$

The vector $\mathbf{E}$ is therefore both irrotational and solenoidal. If S is a closed surface

$$\int_S \mathbf{E} \cdot \mathbf{n}\, da = -e \int_S \mathbf{n} \cdot \nabla_P \frac{1}{r} da = e\omega, \tag{224}$$

where ω is the solid angle subtended by S at Q. Since this solid angle is 4π or zero according as Q lies inside or outside S,

$$\int_S \mathbf{E} \cdot \mathbf{n}\, da = 4\pi e \tag{225}$$

if Q lies inside S and

$$\int_S \mathbf{E} \cdot \mathbf{n}\, da = 0 \tag{226}$$

if Q lies outside.

A hydrodynamic field mathematically identical with the electric field about a point charge may be obtained in the following way. Suppose an amount $4\pi e$ of incompressible fluid to issue per unit time from the source Q and, spreading equally in all directions, flow away. The velocity of this fluid at P will have precisely the value $\mathbf{E}$. That this velocity is irrotational and solenoidal is evident from the nature of the motion. Equations (225) and (226) express that the flow out through a closed surface is equal to the flow in through the included sources, and is zero if there is no interior source (cf. §33).

In case of a system of charges $e_1, e_2, \ldots, e_n$ located at the points $Q_1, Q_2, \ldots, Q_n$, it is assumed that these act independently so that the total intensity at P,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \cdots + \mathbf{E}_n, \tag{227}$$

is the sum of the intensities due to the separate charges. If then S is a surface upon which none of the charges lie,

$$\int_S \mathbf{E} \cdot \mathbf{n}\, da = \int_S \mathbf{E}_1 \cdot \mathbf{n}\, da + \cdots + \int_S \mathbf{E}_n \cdot \mathbf{n}\, da.$$

If S forms the complete boundary of a region V, each integral on the right is 4π times the corresponding charge if that charge is located within V and zero if it is not. Hence

$$\int_S \mathbf{E} \cdot \mathbf{n}\, da = 4\pi \text{ (total charge within } S). \tag{228}$$

The potential at P due to a single charge e is

$$\phi = \int_P^\infty \mathbf{E} \cdot d\mathbf{R} = \int_P^\infty -e\, d\mathbf{R} \cdot \nabla_P \frac{1}{r} = \frac{e}{r}, \tag{229}$$

where r is the distance from the charge to P. Similarly, the potential of a system of charges is

$$\phi = \int_P^\infty (\mathbf{E}_1 + \mathbf{E}_2 + \cdots + \mathbf{E}_n) \cdot d\mathbf{R} = \frac{e_1}{r_1} + \frac{e_2}{r_2} + \cdots + \frac{e_n}{r_n}, \tag{230}$$

where r_1, r_2, etc. are the distances from the charges e_1, e_2, etc. to P. From the general theory or by direct differentiation we find

$$\mathbf{E} = -\nabla\phi = -\left[e_1 \nabla \frac{1}{r_1} + e_2 \nabla \frac{1}{r_2} + \cdots + e_n \nabla \frac{1}{r_n}\right]. \tag{231}$$

54. Space Charges. Suppose a charge is spread continuously over a region V, the density of charge at the point $Q(\xi, \eta, \zeta)$ being $\rho(\xi, \eta, \zeta)$. Cut the region into cells Δv and let Q be a point within one of these cells. Except for infinitesimals of higher

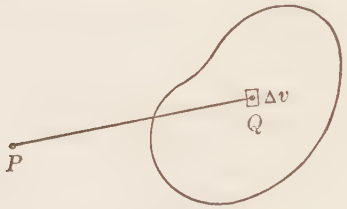


FIG. 51

order than Δv the potential at an outside point $P(x, y, z)$ due to charges within Δv is the same as that of a point charge $\rho \Delta v$ at Q, namely

$$\frac{\rho\, \Delta v}{r},$$

where r is the distance PQ. If then P lies outside the region V,

the potential at P due to the continuous distribution is the limit

$$\phi(x, y, z) = \int_V \frac{\rho\, dv}{r}. \tag{232}$$

Similarly, the electric intensity at P due to charges within Δv is approximately

$$-\rho \nabla_P \frac{1}{r} \Delta v$$

and the total intensity at P is the limit

$$\mathbf{E}(x, y, z) = -\int_V \rho \nabla_P \frac{1}{r} dv. \tag{233}$$

Since P lies outside the charged region, the function $\frac{1}{r}$ and its first derivatives with respect to x, y, z are continuous for all positions of Q within V. Therefore

$$\nabla\phi = \int_V \rho \nabla_P \frac{1}{r} dv$$

can be obtained from (232) by differentiation under the integral sign.* Comparison of this with (233) shows that

$$\mathbf{E} = -\nabla\phi, \tag{234}$$

as we should expect from the general relation between potential and electric intensity.

In the integrals (232) and (233), the point P is kept fixed and the summation taken for all volume elements dv in the region V. The result in each case is a function of the coördinates x, y, z of P. Now let P belong to a closed surface S which has no point in common with V. We wish to evaluate the integral

$$\int_S \mathbf{E} \cdot \mathbf{n}\, da$$

when V lies entirely inside S and when V lies entirely outside. To do this we observe that if V is cut into cells and a charge $\rho\Delta v$ placed at some point Q of each cell Δv, the resulting system

* cf. F. S. Woods, "Advanced Calculus," page 141.

of point charges determines at P the electric intensity

$$\mathbf{E}' = -\sum \rho \nabla_P \frac{1}{r} \Delta v.$$

By (228) the integral of $\mathbf{E}'$ over S is

$$\int_S \mathbf{E}' \cdot \mathbf{n}\, da = 4\pi \sum \rho\, \Delta v$$

if V lies inside S and

$$\int_S \mathbf{E}' \cdot \mathbf{n}\, da = 0$$

if V lies outside. Since S and V have no common points, the cells can be taken so small that $\mathbf{E}'$ differs as little as we please from $\mathbf{E}$ at all points of S. The above equations therefore give as limits

$$\int_S \mathbf{E} \cdot \mathbf{n}\, da = 4\pi \int_V \rho\, dv \tag{235}$$

if V lies entirely inside S and

$$\int_S \mathbf{E} \cdot \mathbf{n}\, da = 0 \tag{236}$$

if entirely outside.

55. Improper Integrals. In the integrals (232) and (233) r is the distance between the element of volume dv and the point P at which the potential or electric intensity is measured. If P lies inside the charged region, $\frac{1}{r}$ becomes infinite at P and these integrals (when defined in the ordinary way as limits of sums) cease to have definite values. In these cases the function integrated becomes infinite at the single point P. In other cases it may become infinite at all points of a curve or surface. But in all cases for which the integral can be defined it is possible to enclose such points in a region of total volume as small as we please.

In such cases the value of the integral

$$\int_V f\, dv \tag{237}$$

(if it has a value) is defined in the following way. Divide V into two parts, the one V' being a region in which f is bounded, the other V'' containing all points at which f becomes infinite. The integral

$$\int_{V'} f\,dv \tag{238}$$

can be defined in the ordinary way. If this integral approaches a definite limit (independent of the shape of V'') when the volume of V'' approaches zero, the integral (237) is said to converge and its value is defined as the limit approached by (238).

The condition for convergence of (237) is that the part

$$\int_{V_0} f\,dv$$

of the integral in any sub-region V_0 shall approach zero as limit when the volume of V_0 approaches zero.

In discussing the convergence of the potential integrals, the following theorems will be found of particular value.

THEOREM I. *Let V be a finite region and r the distance between the element of volume dv and a fixed point P. The integral*

$$\int_V \frac{dv}{r^n} \tag{239}$$

is convergent if n is a positive constant less than 3.

To prove this let S be a sphere of radius a and center P. Any region V_0 can be divided into two parts, the one V_1 being inside S, the other V_2 being outside. Since the element of integration is positive, the integral extended through V_1 is not greater than that extended throughout the whole of S. That is

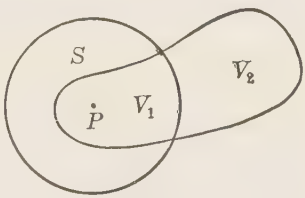


FIG. 52

$$\int_{V_1} \frac{dv}{r^n} \leqq \int_S \frac{dv}{r^n} = 4\pi \int_0^a \frac{r^2\,dr}{r^n} = \left[\frac{4\pi r^{3-n}}{3-n}\right]_0^a$$

If $n < 3$ the function r^{3-n} vanishes at the lower limit and so

$$\int_{V_1} \frac{dv}{r^n} \leqq \frac{4\pi}{3-n} a^{3-n}. \tag{240}$$

Outside the sphere $r > a$ and so

$$\int_{V_2} \frac{dv}{r^n} \leqq \frac{v_2}{a^n} \leqq \frac{v_0}{a^n}, \tag{241}$$

where v_2 and v_0 are the volumes of V_2 and V_0 respectively. If ϵ is any given positive number we can take a so small that

$$\frac{4\pi}{3-n} a^{3-n} < \frac{\epsilon}{2},$$

and can then find a number δ such that

$$\frac{\delta}{a^n} < \frac{\epsilon}{2}.$$

If then V_0 is any region with volume less than δ, by adding (240) and (241) we get

$$\int_{V_0} \frac{dv}{r^n} = \int_{V_1} \frac{dv}{r^n} + \int_{V_2} \frac{dv}{r^n} < \epsilon,$$

whatever shape V_0 may have. The integral is therefore convergent.

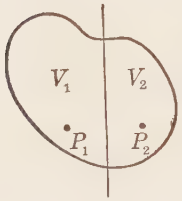


FIG. 53

THEOREM II. *Let r_1, r_2 be the distances between the element of volume dv and the fixed points P_1, P_2 respectively. If m and n are positive constants with sum less than 3 and ϵ is any positive number, a value δ can be found independent of P_1 and P_2 such that*

$$\int_{V_0} \frac{dv}{r_1^m r_2^n} < \epsilon \tag{242}$$

for every region V_0 with volume less than δ.

To prove this construct the plane perpendicular to the line P_1P_2 at its midpoint. This plane divides any region V_0 into two parts, V_1, V_2 on the same side as P_1, P_2 respectively. In the integral through V_1, $r_1 \leqq r_2$ and so

$$\int_{V_1} \frac{dv}{r_1^m r_2^n} \leqq \int_{V_1} \frac{dv}{r_1^{m+n}} \leqq \int_{V_0} \frac{dv}{r_1^{m+n}}. \tag{243}$$

Similarly

$$\int_{V_2} \frac{dv}{r_1^m r_2^n} \leqq \int_{V_2} \frac{dv}{r_2^{m+n}} \leqq \int_{V_0} \frac{dv}{r_2^{m+n}}. \tag{244}$$

The last integrals in (243) and (244) satisfy the conditions of Theorem I. By making V_0 sufficiently small these can both be made less than $\frac{1}{2}\epsilon$. Then

$$\int_{V_0} \frac{dv}{r_1^m r_2^n} = \int_{V_1} \frac{dv}{r_1^m r_2^n} + \int_{V_2} \frac{dv}{r_1^m r_2^n} < \epsilon.$$

The integrals so far considered are three-dimensional. In a similar way improper integrals are handled in the plane. If $f(x, y)$ becomes infinite in the region A of the plane, we divide A into two parts, the one A'' containing the points at which f becomes infinite, the other A' being a region in which f is bounded. If the integral

$$\int_{A'} f\, da \tag{245}$$

approaches a limit when the area of A'' approaches zero (independent of the shape of A'') the integral

$$\int_A f\, da$$

is said to converge and its value is defined as the limit approached by (245).

By methods similar to those used above the following theorems can be proved.

THEOREM III. *Let A be a finite plane area and r the distance between the element of area da and the fixed point P. The integral*

$$\int_A \frac{da}{r^n}$$

is convergent if n is a positive constant less than 2.

THEOREM IV. *Let r_1, r_2 be the distances between the element of area da and the fixed points P_1, P_2. If m and n are positive constants with sum less than 2 and ϵ is any positive number, a value δ can be found independent of P_1 and P_2 such that*

$$\int_{A_0} \frac{da}{r_1^m r_2^n} < \epsilon$$

for every region A_0 with area less than δ.

In proving the continuity of functions represented by improper integrals we shall make frequent use of the following theorem.

THEOREM V. *A function $\phi(P)$ is continuous at P_0 if to each positive number ϵ there corresponds a division of $\phi(P)$ into two parts*

$$\phi(P) = \phi'(P) + \phi''(P),$$

such that $\phi'(P)$ is continuous at P_0 and

$$|\phi''(P)| < \epsilon$$

for all positions of P in a sufficiently small sphere of center P_0.

To prove this let $\phi'(P)$, $\phi''(P)$ be the functions corresponding to the positive number $\frac{1}{3}\epsilon$. There will then be a sphere S' of center P_0 within which

$$|\phi'(P) - \phi'(P_0)| < \frac{\epsilon}{3}$$

and a sphere S'' of center P_0 within which

$$|\phi''(P)| < \frac{\epsilon}{3}.$$

Let S be the smaller of these spheres. For any point P in S we then have

$$\phi(P) = \phi'(P) + \phi''(P),$$
$$\phi(P_0) = \phi'(P_0) + \phi''(P_0),$$
$$|\phi(P) - \phi(P_0)| \leqq |\phi'(P) - \phi'(P_0)| + |\phi''(P)| + |\phi''(P_0)| < \epsilon,$$

which proves the continuity of ϕ at P_0.

56. Values Inside the Charged Region. We have defined the electric intensity at P as the force exerted by the field on a unit positive charge placed at P. If P lies inside the charged region this definition is not applicable. For the space charge near P must be removed before a new charge can be introduced. And if we do this we measure not the original field but a different one. A new definition is therefore required. The one generally used is the following.

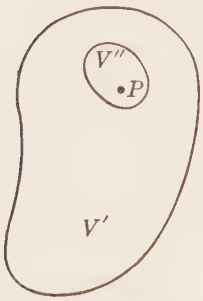


FIG. 54

Remove the space charge from a small region V'' having P as an interior point. Determine the electric intensity $\mathbf{E}'$ at P due to the remaining space charges. If this value $\mathbf{E}'$ approaches a definite limit when the volume of V'' approaches zero (independent of the shape of V'') that limit is defined as

the electric intensity at P. If $\mathbf{E}$ is expressed as a volume integral, the condition that the limit exist is identical with the condition that the integral converge. In a similar way we define potential or any other physical quantity at P.

If ρ is a bounded integrable function we shall now show that *the expressions*

$$\phi = \int_V \frac{\rho\, dv}{r}, \tag{246}$$

$$\mathbf{E} = -\int_V \rho \nabla_P \frac{1}{r}\, dv, \tag{247}$$

which represent potential and electric intensity at points outside the space charge, are also valid at points inside the charge. According to the above definitions all that is required is to show that the integrals (246) and (247) converge. Since ρ is bounded there is a number m such that

$$|\rho| < m$$

at all points of V. Consequently

$$\left|\frac{\rho}{r}\right| < \frac{m}{r}.$$

Also, since ∇r is a unit vector,

$$\left|\rho \nabla_P \frac{1}{r}\right| = \left|-\rho \frac{\nabla r}{r^2}\right| < \frac{m}{r^2}.$$

If then V_0 is any small region belonging to V,

$$\left|\int_{V_0} \frac{\rho\, dv}{r}\right| < m \int_{V_0} \frac{dv}{r},$$

$$\left|\int_{V_0} \rho \nabla_P \frac{1}{r}\, dv\right| < m \int_{V_0} \frac{dv}{r^2}.$$

By Theorem I, §55, these integrals can both be made as small as we please by making V_0 sufficiently small. Hence (246) and (247) both converge.

Next we show that ϕ *and* $\mathbf{E}$ *are continuous and satisfy the equation*

$$\mathbf{E} = -\nabla\phi \tag{248}$$

inside as well as outside the space charge. To prove the continuity at a point P_0, let V'' be a small sphere of center P_0 and V' the remainder of the charged region. The potential at any point P is

$$\phi = \int_V \frac{\rho\, dv}{r} = \int_{V'} \frac{\rho\, dv}{r} + \int_{V''} \frac{\rho\, dv}{r}.$$

Since the integrals are convergent, V'' can be taken so small that

$$\phi'' = \int_{V''} \frac{\rho\, dv}{r}$$

is in numerical value less than a given positive number ϵ for all positions of P. Since P_0 lies outside the region V'

$$\phi' = \int_{V'} \frac{\rho\, dv}{r}$$

is continuous at P_0. Thus

$$\phi = \phi' + \phi''$$

is continuous at P_0 since it is the sum of two parts, one of which is continuous, the other numerically less than ϵ in a region about P_0 (Theorem V, §55). In a similar way we prove the continuity of **E**.

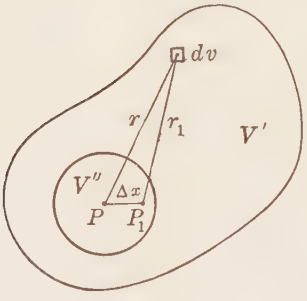


FIG. 55

In view of the expressions (246) and (247) for ϕ and **E**, equation (248) expresses that first derivatives of ϕ can be obtained by differentiation under the integral sign. To show that this is the case it will be sufficient to consider the derivative with respect to x as the other derivatives can be handled in the same way. Let V'' be a small sphere of center $P(x, y, z)$ and let V' be the remainder of the charged region. The potential at P is

$$\phi(x, y, z) = \int_V \frac{\rho\, dv}{r} = \int_{V'} \frac{\rho\, dv}{r} + \int_{V''} \frac{\rho\, dv}{r}.$$

If $P_1(x + \Delta x, y, z)$ is a neighboring point, having the same y and z as P, and r_1 is its distance from dv, the potential at P_1 is

$$\phi(x + \Delta x, y, z) = \int_V \frac{\rho\, dv}{r_1} = \int_{V'} \frac{\rho\, dv}{r_1} + \int_{V''} \frac{\rho\, dv}{r_1}.$$

The increment in potential when x changes to $x + \Delta x$ is the difference of these values. Dividing this by Δx we have

$$\frac{\Delta\phi}{\Delta x} = \int_{V'} \frac{\rho\left(\frac{1}{r_1} - \frac{1}{r}\right)}{\Delta x} dv + \int_{V''} \frac{\rho\left(\frac{1}{r_1} - \frac{1}{r}\right)}{\Delta x} dv. \tag{249}$$

Since the difference of two sides of a triangle is not greater than the third

$$|r - r_1| \leqq \Delta x, \qquad \left|\frac{\partial r}{\partial x}\right| \leqq 1,$$

and so

$$\left|\int_{V''} \frac{\rho\left(\frac{1}{r_1} - \frac{1}{r}\right)}{\Delta x} dv\right| = \left|\int_{V''} \rho \frac{r - r_1}{r r_1 \Delta x} dv\right| \leqq m \int_{V''} \frac{dv}{r r_1}, \tag{250}$$

$$\left|\int_{V''} \rho \frac{\partial}{\partial x}\left(\frac{1}{r}\right) dv\right| = \left|\int_{V''} \frac{\rho}{r^2} \frac{\partial r}{\partial x} dv\right| \leqq m \int_{V''} \frac{dv}{r^2}. \tag{251}$$

By Theorems I, II, §55, we can take V'' so small that these integrals are both less than $\frac{1}{3}\epsilon$. Since P lies outside V' we can then take Δx so small that the first integral in (249) differs from its limit

$$\int_{V'} \rho \frac{\partial}{\partial x}\left(\frac{1}{r}\right) dv$$

by less than $\frac{1}{3}\epsilon$. Also (251) shows that this limit differs from

$$\int_{V} \rho \frac{\partial}{\partial x}\left(\frac{1}{r}\right) dv \tag{252}$$

by less than $\frac{1}{3}\epsilon$. For such values of Δx the ratio $\frac{\Delta\phi}{\Delta x}$ differs from (252) by less than ϵ. Since ϵ can be made as small as we please it follows that

$$\lim_{\Delta x \to 0} \frac{\Delta\phi}{\Delta x} = \int_{V} \rho \frac{\partial}{\partial x}\left(\frac{1}{r}\right) dv$$

which was to be proved.

Finally we shall show that *equation* (235) *is valid for a surface* S *which cuts through the space charge.* To do this let V'' be a region containing all parts of the space charge within a distance δ from S and let $\mathbf{E}'$ be the electric intensity due to the remainder

of the charge. By (235)

$$\int_S \mathbf{E}' \cdot \mathbf{n}\, da = 4\pi \int \rho\, dv,$$

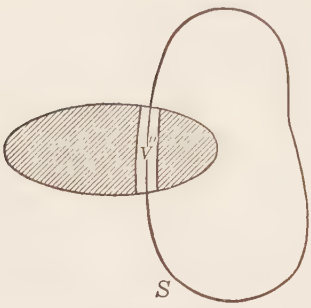


Fig. 56

the integral on the right being extended through the space charges inside S but outside V''. By making the volume of V'' sufficiently small we can make the difference $|\mathbf{E} - \mathbf{E}'|$ as small as we please at all points of space. When V'' approaches zero the above equation therefore gives as limit

$$\int_S \mathbf{E} \cdot \mathbf{n}\, da = 4\pi \int \rho\, dv, \quad (253)$$

the integral on the right being taken over the region inside S.

57. Higher Derivatives of Potential. For derivatives of the potential higher than the first to exist and be continuous, the density of charge must be further restricted. For the present it will be sufficient to show that *the second derivatives of ϕ exist and are continuous at interior points of a region where ρ and its first derivatives are continuous.*

To show this write the electric intensity in the form

$$\mathbf{E} = \int_V \rho \nabla_Q \frac{1}{r} dv = \int_V \nabla\left(\frac{\rho}{r}\right) dv - \int_V \frac{\nabla \rho}{r} dv.$$

By (124)

$$\int_V \nabla\left(\frac{\rho}{r}\right) dv = \int_S \frac{\rho}{r} \mathbf{n}\, da,$$

where S is the surface forming the complete boundary of V. Hence

$$\mathbf{E} = \int_S \frac{\rho}{r} \mathbf{n}\, da - \int_V \frac{\nabla \rho}{r} dv.$$

At an interior point of V the value of r in the first integral does not become zero and so that integral has continuous derivatives of all orders. The second integral is

$$\int_V \frac{\nabla \rho}{r} dv = \mathbf{i} \int_V \frac{1}{r} \frac{\partial \rho}{\partial x} dv + \mathbf{j} \int_V \frac{1}{r} \frac{\partial \rho}{\partial y} dv + \mathbf{k} \int_V \frac{1}{r} \frac{\partial \rho}{\partial z} dv.$$

The coefficients of **i**, **j**, **k** are integrals of the same form as

$$\phi = \int \frac{\rho}{r} dv,$$

which we have shown has continuous first derivatives. Hence the components of **E** have continuous first derivatives. But these are the second derivatives of ϕ. Therefore the second derivatives of ϕ exist and are continuous.

In a similar way we can show that if ρ and all its derivatives of order less than n are continuous, then ϕ and all its derivatives of order less than $n + 1$ are continuous. It should be noted however that these may not be obtained by simple differentiation under the integral sign. Thus from

$$\phi = \int \frac{\rho}{r} dv$$

we cannot conclude that

$$\frac{\partial^2 \phi}{\partial x^2} = \int \rho \frac{\partial^2 \left(\frac{1}{r}\right)}{\partial x^2} dv.$$

The last equation may not be correct.

58. Laplace's and Poisson's Equations. Let **E** and its first derivatives be continuous at all points of a region V. According to the preceding section this is true if ρ has continuous first derivatives. If V_1 is any part of V and S_1 is the surface forming its complete boundary, by the divergence theorem

$$\int_{S_1} \mathbf{E} \cdot \mathbf{n}\, da = \int_{V_1} \nabla \cdot \mathbf{E}\, dv.$$

Equation (253) shows that this is equivalent to

$$\int_{V_1} \nabla \cdot \mathbf{E}\, dv = 4\pi \int_{V_1} \rho\, dv.$$

Since this holds for every region V_1 belonging to V,

$$\nabla \cdot \mathbf{E} = 4\pi\rho. \tag{254}$$

If **E** is replaced by its value $-\nabla\phi$, this becomes

$$\nabla^2 \phi = -4\pi\rho \tag{255}$$

which is called **Poisson's equation.**

In this equation ρ is the density of charge at the point where

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

is calculated. At a point outside the charge ρ is zero and (255) is replaced by

$$\nabla^2\phi = 0, \tag{256}$$

which is known as **Laplace's equation.**

If r is the distance between $P(x, y, z)$ and any fixed point,

$$\phi = \frac{1}{r}$$

is the potential at P due to a unit charge at that point. We should therefore expect this function to be a solution of Laplace's equation. That

$$\nabla^2\frac{1}{r} = 0 \tag{257}$$

can be shown by performing the differentiations. Since

$$\nabla^2\left[\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right] = \frac{\partial}{\partial x}\nabla^2\frac{1}{r} = 0,$$

it follows that the partial derivatives of $\frac{1}{r}$ with respect to x, y, z are also solutions of Laplace's equation.

59. Hölder Condition.* We have just shown that ϕ satisfies Poisson's equation at a point P if ρ and its first derivatives are continuous in a small neighborhood about P. Because of the great importance of this equation we shall prove its validity under less restricted conditions.

Let r_0 be the distance P_0Q. If three positive constants a, n, c exist such that

$$|\rho(Q) - \rho(P_0)| < ar_0^n \tag{258}$$

for all points Q for which $r_0 < c$, the charge density ρ is said to satisfy a Hölder condition at P_0.

A function ρ which is differentiable at P_0 satisfies (258) with $n = 1$. A Hölder condition may however be satisfied at a point where ρ is not differentiable either because its variation

* cf. O. D. Kellogg, "Potential Theory," page 152.

with distance is too rapid or because its variations in different directions are not properly related. In particular a function satisfies a Hölder condition at a point where it is continuous and in a neighborhood about which it is piecewise differentiable. For the ratio

$$\frac{\rho(Q) - \rho(P_0)}{r_0}$$

is less than some number m for points Q in this neighborhood. If $n = 1$ and a is any positive number greater than m, the condition (258) is satisfied.

At a point where ρ satisfies a Hölder condition, the potential satisfies Poisson's equation.

To prove this let P_0 be such a point, ρ_0 the charge density at P_0, and

$$\psi_0 = \int_V \frac{\rho_0 \, dv}{r} \tag{259}$$

the potential at P due to a distribution of constant density ρ_0 spread over V. Consider the difference

$$\psi = \phi - \psi_0 = \int_V \frac{\rho - \rho_0}{r} dv$$

between this and the actual potential at P. In this integral

$$r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$$

is the distance between $P(x, y, z)$ and the point $Q(\xi, \eta, \zeta)$ at which the element of integration is located. The first derivative of ψ with respect to x thus has the value

$$\psi_x(x_0, y_0, z_0) = -\int_V (\rho - \rho_0) \frac{x_0 - \xi}{r_0^3} dv \tag{260}$$

at $P_0(x_0, y_0, z_0)$ and the value

$$\psi_x(x, y_0, z_0) = -\int_V (\rho - \rho_0) \frac{x - \xi}{r^3} dv$$

at a neighboring point $P(x, y_0, z_0)$. To obtain the second derivative of ψ at P_0 we form the quotient

$$\frac{\psi_x(x, y_0, z_0) - \psi_x(x_0, y_0, z_0)}{x - x_0}$$

$$= \int_V \frac{\rho - \rho_0}{x - x_0} \left[(x - \xi) \left(\frac{1}{r_0^3} - \frac{1}{r^3} \right) - \frac{x - x_0}{r_0^3} \right] dv. \tag{261}$$

Since

$$x - \xi \leqq r, \qquad |r - r_0| \leqq |x - x_0|,$$

$$\frac{1}{r_0^3} - \frac{1}{r^3} = \frac{r - r_0}{rr_0}\left(\frac{1}{r_0^2} + \frac{1}{rr_0} + \frac{1}{r^2}\right),$$

the expression to be integrated is not greater than

$$\frac{|\rho - \rho_0|}{r_0}\left(\frac{2}{r_0^2} + \frac{1}{rr_0} + \frac{1}{r^2}\right).$$

From (258) this is less than

$$a\left(\frac{2}{r_0^{3-n}} + \frac{1}{rr_0^{2-n}} + \frac{1}{r^2 r_0^{1-n}}\right),$$

each term of which satisfies the conditions of Theorem II, §55. If then we construct a sphere V'' of center P_0, we can take the sphere so small that the part of (261) in V'' is numerically less than a given positive number ϵ for all values of $\Delta x = x - x_0$, while the remaining part in V' has the limit

$$\int_{V'} (\rho - \rho_0)\left[\frac{3(x_0 - \xi)^2}{r_0^5} - \frac{1}{r_0^3}\right] dv,$$

obtained by differentiating (260) under the integral sign with respect to x_0. Since

$$|x_0 - \xi| \leqq r_0,$$

a discussion similar to that above shows that this integral converges when V'' approaches zero. Hence

$$\frac{\partial^2 \psi}{\partial x^2} = \int_V (\rho - \rho_0)\left[\frac{3(x_0 - \xi)^2}{r_0^5} - \frac{1}{r_0^3}\right] dv$$

is the second derivative with respect to x at P_0. In a similar way we show that the second derivatives with respect to y and z at P_0 exist and have the values

$$\frac{\partial^2 \psi}{\partial y^2} = \int_V (\rho - \rho_0)\left[\frac{3(y_0 - \eta)^2}{r_0^5} - \frac{1}{r_0^3}\right] dv,$$

$$\frac{\partial^2 \psi}{\partial z^2} = \int_V (\rho - \rho_0)\left[\frac{3(z_0 - \zeta)^2}{r_0^5} - \frac{1}{r_0^3}\right] dv.$$

Adding these equations and using the relation

$$(x_0 - \xi)^2 + (y_0 - \eta)^2 + (z_0 - \zeta)^2 = r_0^2,$$

we obtain

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0.$$

Now

$$\phi = \psi + \psi_0.$$

Since ρ_0 is constant, it is continuously differentiable, and so

$$\nabla^2\psi_0 = -4\pi\rho_0.$$

Hence

$$\nabla^2\phi = \nabla^2\psi + \nabla^2\psi_0 = -4\pi\rho_0$$

at P_0, which was to be proved.

60. Surface Charges. Let a charge be located upon a surface S, the surface density (charge per unit area) at $Q(\xi, \eta, \zeta)$ being $\sigma(\xi, \eta, \zeta)$. We assume that σ is continuous, or piecewise continuous, on S.

The potential at an outside point $P(x, y, z)$ due to the charge in the area Δa about Q is approximately

$$\frac{\sigma\,\Delta a}{r},$$

where r is the distance PQ, and the total potential at P is the limit

$$\phi = \int_S \frac{\sigma\,da}{r}. \tag{262}$$

The electric intensity at P due to the surface charge is

$$\mathbf{E} = -\nabla\phi = -\int_S \sigma\,\nabla_P\frac{1}{r}\,da = \int_S \sigma\,\nabla\frac{1}{r}\,da. \tag{263}$$

At points not belonging to the charged surface it is evident from these equations that ϕ and $\mathbf{E}$ are continuous functions of x, y, z. Since

$$\nabla^2\frac{1}{r} = 0$$

it follows further that

$$\nabla^2\phi = 0$$

and so ϕ is a solution of Laplace's equation.

61. Region Near a Point on a Surface. In discussing values near a point Q_0 of a surface at which there is a tangent plane we

shall take Q_0 as origin, the normal at Q_0 as z-axis, and use the letters P', Q', r', s to represent the projections of P, Q, $r = PQ$, and $r_0 = Q_0Q$ on the xy-plane. Let S'' be a small region on the

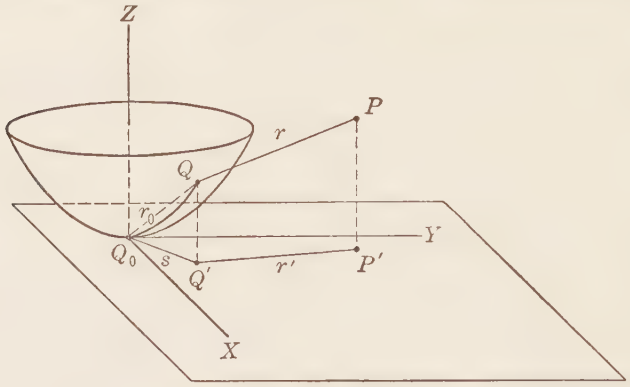


FIG. 57

surface having Q_0 as an interior point and let S' be the remainder of S.

At points near Q_0 the angle γ between the z-axis and the normal at Q is nearly zero. If $c > 1$ we can then take S'' so small that

$$\sec \gamma < c. \tag{264}$$

The curves x = constant, y = constant determine on the surface an element of area

$$da = \sec \gamma \, da'$$

which has

$$da' = dx \, dy$$

for projection on the xy-plane. From (264) we then have

$$da < c \, da'. \tag{265}$$

At points near Q_0 the unit normal n is a function of x and y. If Q_0 is an interior point of a region S'' in which this function is continuously differentiable, the surface is said to be continuously curved, or to have continuous curvature, at Q_0. If this region is closed the partial derivatives of n have a maximum value and so there is a positive number m such that

$$\left(\frac{\partial \mathrm{n}}{\partial x}\right)^2 + \left(\frac{\partial \mathrm{n}}{\partial y}\right)^2 < m^2 \tag{266}$$

for all points Q in S''. If $\mathbf{n}$ is the normal at Q and $\mathbf{n}_0$ the normal at Q_0 we have

$$\mathbf{n} - \mathbf{n}_0 = \int_{Q_0}^{Q} d\mathbf{n}.$$

We can take this integral along the curve which has Q_0Q' for projection on the xy-plane. Then

$$d\mathbf{n} = \frac{\partial \mathbf{n}}{\partial x}dx + \frac{\partial \mathbf{n}}{\partial y}dy = \left(\frac{\partial \mathbf{n}}{\partial x}\cos\theta + \frac{\partial \mathbf{n}}{\partial y}\sin\theta\right)ds,$$

where ds is the differential distance along Q_0Q' and θ is the angle between that line and the x-axis. Together with (266) this gives $|d\mathbf{n}| < m\,ds$ and so

$$|\mathbf{n} - \mathbf{n}_0| < ms. \tag{267}$$

If three positive constants a, n, c exist such that

$$|\sigma(Q) - \sigma(Q_0)| < ar_0^n \tag{268}$$

for every point Q at a distance r_0 less than c from Q_0 the charge density σ is said to satisfy a Hölder condition at Q_0 (cf. §59). If Q_0 is a regular point the ratio of r_0 to s will have a finite maximum in a region $r_0 \leqq \delta$ and so

$$|\sigma(Q) - \sigma(Q_0)| < bs^n, \tag{269}$$

where b is another positive constant.

If the condition (268) with fixed values of a, n, and c holds for each point Q_0 in a certain region of the surface, the charge density σ is said to satisfy a uniform Hölder condition in that region.

62. Potential at Points on the Charged Surface. At a point on the charged surface r becomes zero and the integrals representing potential and electric intensity are improper. If σ is bounded the potential is however definite and continuous even at such points.

To show this suppose as a first case that Q_0 is a regular point on the surface. Use the notation of §61. Let S'' be a small region about Q_0 and A'' its projection on the xy-plane. The potential at $P(x, y, z)$ due to the charge on S'' is

$$\phi'' = \int_{S''} \frac{\sigma\, da}{r}.$$

Since $r \geqq r'$, by (265)

$$|\phi''| < c\mu \int_{A''} \frac{da'}{r'},$$

where μ is the maximum value of σ. In this integral r' is the distance from da' to P'. By making S'' sufficiently small $|\phi''|$ can be made less than ϵ for all positions of P' (Theorem III, §55). The potential

$$\phi' = \int_{S'} \frac{\sigma\, da}{r}$$

due to the remainder of the charge is continuous at Q_0. Hence

$$\phi = \phi' + \phi''$$

is also continuous.

If Q_0 is located at the junction of two or more regular surface elements, the surface can be cut into a finite number of parts to each of which the above argument applies.

63. Electric Intensity at Points Near the Surface. At points on the charged surface the electric intensity is usually not continuous. When the point $P(x, y, z)$ at which $\mathbf{E}$ is measured approaches a point Q_0 on the surface, the limit of $\mathbf{E}$ depends on the side of the surface from which the approach is made. And even when the approach is from one side a limit may not exist if the charge density and nature of the surface near Q_0 fail to satisfy certain conditions.

We shall consider only the case of a surface S which is continuously curved and on which σ satisfies a uniform Hölder condition (cf. §61). At an interior point Q_0 of such a surface we shall prove that the electric intensity has definite limits $\mathbf{E}_+$, $\mathbf{E}_-$ depending on whether the approach is from the positive or the negative side of the surface, and that these limits are continuous functions of Q_0 which satisfy the equation

$$\mathbf{E}_+ - \mathbf{E}_- = 4\pi\sigma_0\mathbf{n}_0, \tag{270}$$

where σ_0 is the charge density and $\mathbf{n}_0$ the unit normal at Q_0.

To prove this let S'' be a small region having Q_0 as interior point and let S' be the remainder of S. If P is sufficiently near Q_0 there will be a normal at some point Q_1 of S'' passing through

P. If there are several such points Q_1 take one as near P as possible. With P associate the vector

$$\mathbf{F}(P) = \mathbf{E}(P) + \sigma_1 \mathbf{n}_1 \omega(P),$$

where ω is the solid angle subtended by S at P and σ_1, $\mathbf{n}_1$ are charge density and unit normal at Q_1. We shall show that $\mathbf{F}$ is continuous at Q_0. Since σ_1, $\mathbf{n}_1$, $\omega(P)$ have definite and continuous limits on either side of the surface, the same is then true for $\mathbf{E}$. Equating the limits $\mathbf{F}_+$, $\mathbf{F}_-$ at Q_0 and using equation (218) we get (270).

It remains then only to show that $\mathbf{F}$ is continuous at Q_0. Expressing $\mathbf{E}(P)$ and $\omega(P)$ as integrals we have

$$\mathbf{F} = \int_S \left(\sigma \nabla \frac{1}{r} - \mathbf{n}_1 \sigma_1 \mathbf{n} \cdot \nabla \frac{1}{r} \right) da.$$

This is the sum of three integrals

$$\mathbf{F}_1 = \int_S \sigma_1 \left[\nabla \frac{1}{r} - \mathbf{n} \left(\mathbf{n} \cdot \nabla \frac{1}{r} \right) \right] da,$$

$$\mathbf{F}_2 = \int_S (\sigma - \sigma_1) \nabla \frac{1}{r} da,$$

$$\mathbf{F}_3 = \int_S \sigma_1 (\mathbf{n} - \mathbf{n}_1) \mathbf{n} \cdot \nabla \frac{1}{r} da.$$

From the identity

$$(\mathbf{n} \times \nabla) \times \frac{\mathbf{n}}{r} - \frac{(\mathbf{n} \times \nabla) \times \mathbf{n}}{r} = \nabla \frac{1}{r} - \mathbf{n} \left(\mathbf{n} \cdot \nabla \frac{1}{r} \right)$$

and (123) we obtain

$$\mathbf{F}_1 = \sigma_1 \int_C d\mathbf{R} \times \frac{\mathbf{n}}{r} - \sigma_1 \int_S \frac{(\mathbf{n} \times \nabla) \times \mathbf{n}}{r} da,$$

where C is the curve forming the boundary of S. The first integral is continuous since r does not become zero. The second is continuous because each of its components is an integral of the same type as the potential of a surface charge. To investigate the integrals $\mathbf{F}_2$ and $\mathbf{F}_3$ express each as the sum of two parts

$$\mathbf{F}_2 = \mathbf{F}_2' + \mathbf{F}_2'', \quad \mathbf{F}_3 = \mathbf{F}_3' + \mathbf{F}_3'',$$

taken over S' and S'' respectively. For points P near Q_0 the parts $\mathbf{F}_2'$, $\mathbf{F}_3'$ are continuous. To evaluate the other parts take

the tangent plane at Q_1 as xy-plane and use the notation of §61 with the subscript unity replacing zero in all the equations. Since P lies on the normal

$$r \geqq s, \qquad \left|\nabla \frac{1}{r}\right| \leqq \frac{1}{s^2}.$$

By use of this relation and (265), (267), (269) we get

$$|\mathbf{F}_2''| < bc \int_{A''} \frac{da'}{s^{2-n}}, \qquad |\mathbf{F}_3''| < mc\,|\sigma_1| \int_{A''} \frac{da'}{s},$$

where A'' is the projection of S'' on the tangent plane. By Theorem III, §55, both of these can be made as small as we please by making S'' sufficiently small. At Q_0 the vector $\mathbf{F}$ is thus continuous since it can be expressed as a sum of parts which are either continuous or else can be made as small as we please by taking S'' sufficiently small (Theorem V, §55).

64. Double Layer. Let $\mathbf{U}$ be the vector from a charge $-e$ to a numerically equal charge $+e$. Suppose the charges approach coincidence at a point Q and at the same time increase in magnitude so that the product

$$e\mathbf{U} = \mathbf{M} \tag{271}$$

remains constant. The resulting limit is called a **doublet** and the vector $\mathbf{M}$ is called its **moment.**

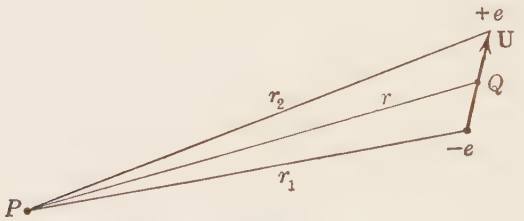


Fig. 58

If r_1, r_2 are the distances from $-e$, e to P, the potential at P due to these charges is

$$e\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

When $\mathbf{U}$ is very small this becomes approximately (cf. equation (54))

$$e\mathbf{U} \cdot \nabla_Q \frac{1}{r} = \mathbf{M} \cdot \nabla_Q \frac{1}{r}.$$

Hence the potential at P due to the doublet is

$$\phi = \mathbf{M} \cdot \nabla_Q \frac{1}{r}. \tag{272}$$

A continuous distribution of doublets on a surface with moments directed along the normal at each point is called a **double layer.** Let $\tau(\xi, \eta, \zeta)$ be a number equal in magnitude to the moment per unit area at $Q(\xi, \eta, \zeta)$ and positive or negative according as the moment is directed positively or negatively along the normal. The total moment of the doublets within an element of area da is then

$$\tau \mathbf{n}\, da$$

and the potential of the double layer at an outside point $P(x, y, z)$ is

$$\phi(x, y, z) = \int_S \tau \mathbf{n} \cdot \nabla \frac{1}{r} da, \tag{273}$$

where S is the surface over which the double layer is spread and r is the distance from the element of area da to P. Since

$$d\omega = -\mathbf{n} \cdot \nabla_Q \frac{1}{r} da$$

is the solid angle subtended at P by the directed element of area $\mathbf{n}\, da$, the potential can also be written

$$\phi = -\int_S \tau\, d\omega. \tag{274}$$

In a region which does not reach the double layer ϕ is evidently continuous. Since it contains x, y, z only in partial derivatives of $\frac{1}{r}$, it satisfies Laplace's equation.

At a point Q_0 where τ is continuous and the surface has continuous curvature, ϕ has definite limits ϕ_+, ϕ_- depending on whether the approach is from the positive or the negative side of the surface, and these limits satisfy the equation

$$\phi_+ - \phi_- = 4\pi\tau_0, \tag{275}$$

where τ_0 is the value of τ at Q_0.

To show this consider the function

$$\psi = \phi + \omega\tau_0 = \int_S (\tau_0 - \tau) d\omega,$$

where ω is the solid angle subtended by S at P. This is the sum of two integrals

$$\int_{S''}(\tau_0 - \tau)d\omega + \int_{S'}(\tau_0 - \tau)d\omega$$

taken respectively over a small region S'' about Q_0 and over the remainder of S. The first can be made as small as we please by making S'' sufficiently small. The second is continuous at Q_0. Hence ψ is continuous at Q_0, and since ω has definite limits, ϕ does also. By equating the values ψ_+, ψ_- equation (275) is obtained.

At a point $P(x, y, z)$ which does not belong to S the electric intensity

$$\mathbf{E} = -\nabla\phi$$

is a continuously differentiable function of x, y, z. Since $\mathbf{n}$ is not a function of x, y, z, from equations (63), (64), (65), and (257) we get

$$\nabla_P\left(\mathbf{n}\cdot\nabla_Q\frac{1}{r}\right) + \nabla_P\times\left(\mathbf{n}\times\nabla_Q\frac{1}{r}\right) = 0.$$

If then we write

$$\mathbf{A} = \int_S \tau\mathbf{n}\times\nabla\frac{1}{r}da \tag{276}$$

we have

$$\nabla\phi + \nabla\times\mathbf{A} = 0,$$

whence

$$\mathbf{E} = -\nabla\phi = \nabla\times\mathbf{A}. \tag{277}$$

The quantity $\mathbf{A}$ is called the vector potential of the double layer. *The electric intensity associated with a double layer can thus be expressed either as the negative gradient of a scalar potential or as the curl of a vector potential.*

Suppose Q_0 is an interior point of a region S'' on S in which the surface has continuous curvature and τ has continuous first and second derivatives. The part of the vector potential relating to S'' is

$$\mathbf{A}'' = \int_{S''}\tau\mathbf{n}\times\nabla\frac{1}{r}da = \int_{S''}\mathbf{n}\times\nabla\frac{\tau}{r}da - \int_{S''}\frac{\mathbf{n}\times\nabla\tau\,da}{r}.$$

If C is the curve forming the boundary of S'', the first integral on the right is equivalent to (equation (122))

$$\int_C \frac{\tau\, d\mathbf{R}}{r}$$

which is continuously differentiable at Q_0. If we write

$$\mathbf{n} \times \nabla\tau = \mathbf{i}\sigma_1 + \mathbf{j}\sigma_2 + \mathbf{k}\sigma_3 \tag{278}$$

the second integral,

$$\int_{S''} \frac{\mathbf{n} \times \nabla\tau\, da}{r} = \mathbf{i}\int_{S''} \frac{\sigma_1\, da}{r} + \mathbf{j}\int_{S''} \frac{\sigma_2\, da}{r} + \mathbf{k}\int_{S''} \frac{\sigma_3\, da}{r},$$

having components of the same form as the potential of a surface charge, is also continuous. Hence $\mathbf{A}''$, and consequently $\mathbf{A}$, is continuous at Q_0.

For the same reason $\mathbf{E}$ has definite limits on both sides of the surface. The change in $\mathbf{E}$ when P crosses the surface is the same as the change in $\nabla \times \mathbf{A}''$ and so the same as the change in

$$\mathbf{i} \times \nabla \int_{S''} \frac{\sigma_1\, da}{r} + \mathbf{j} \times \nabla \int_{S''} \frac{\sigma_2\, da}{r} + \mathbf{k} \times \nabla \int_{S''} \frac{\sigma_3\, da}{r}.$$

Since

$$-\nabla \int_{S''} \frac{\sigma_1\, da}{r}$$

is the electric intensity due to a surface charge σ_1, it takes the increment (cf. equation (270))

$$4\pi \mathbf{n}\sigma_1$$

when P crosses from the negative to the positive side. The increment of $\mathbf{E}$ is then

$$-\mathbf{i} \times 4\pi\mathbf{n}\sigma_1 - \mathbf{j} \times 4\pi\mathbf{n}\sigma_2 - \mathbf{k} \times 4\pi\mathbf{n}\sigma_3.$$

Since the surface gradient $\nabla\tau$ in (278) is perpendicular to $\mathbf{n}$, the result is equivalent to

$$\Delta\mathbf{E} = \mathbf{E}_+ - \mathbf{E}_- = -4\pi\, \nabla\tau. \tag{279}$$

Since this vector is tangent to the surface the normal component of $\mathbf{E}$ has the same limit on both sides. If ds is the element of arc along a direction $d\mathbf{R}$ tangent to the surface

$$\Delta\mathbf{E} \cdot \frac{d\mathbf{R}}{ds} = -4\pi \frac{d\tau}{ds}. \tag{280}$$

The component of **E** along any direction tangent to the surface thus has a discontinuity equal to -4π times the derivative of τ in that direction.

65. Two-Dimensional Fields. A field in which the quantities considered are functions of two rectangular coördinates is called two-dimensional. A magnetic field of this type is determined by currents flowing in long straight wires perpendicular to a plane. In such a case the various functions contain only two coördinates and so the field can be interpreted as existing in a plane.

As an electrostatic example consider a space charge of density $\rho(\xi, \eta)$ in a cylinder with generators parallel to the z-axis, intersecting the xy-plane in a finite area A and extending up and down to infinity. We shall determine the electric intensity at a point $P(x, y, 0)$ of the xy-plane.

To do this let r be the distance between $P(x, y, 0)$ and $Q(\xi, \eta, \zeta)$,

$$r' = \sqrt{(x - \xi)^2 + (y - \eta)^2}$$

its projection on the xy-plane, and

$$\mathbf{R}' = \mathbf{i}(x - \xi) + \mathbf{j}(y - \eta)$$

the vector from the projection of Q to P. At P the electric intensity has the value

$$\mathbf{E} = \int \rho \nabla \left(\frac{1}{\sqrt{r'^2 + \zeta^2}} \right) dv = \int \rho \frac{\mathbf{R}' - \mathbf{k}\zeta}{(r'^2 + \zeta^2)^{3/2}} dv,$$

the integrals extending throughout the infinite cylinder. To show that these integrals converge and find their value we first consider the integral

$$\int\int\int \rho \frac{\mathbf{R}' - \mathbf{k}\zeta}{(r'^2 + \zeta^2)^{3/2}} d\xi \, d\eta \, d\zeta$$

for the finite cylinder extending from $\zeta = -h_1$ to $\zeta = h_2$. Integrating with respect to ζ we get

$$\int\int \rho \left[\frac{\mathbf{R}'\zeta}{r'^2 \sqrt{r'^2 + \zeta^2}} + \frac{\mathbf{k}}{\sqrt{r'^2 + \zeta^2}} \right]_{-h_1}^{h_2} d\xi \, d\eta.$$

When h_1 and h_2 increase indefinitely the second term in this integral approaches zero and the first approaches

$$\mathbf{E} = \int_A 2\rho \frac{\mathbf{R}'}{r'^2} da. \tag{281}$$

This can be regarded as the electric intensity at P due to a surface charge of density

$$\sigma(\xi, \eta) = 2\rho(\xi, \eta), \tag{282}$$

spread over the area A, this charge exerting upon the unit charge at P a force inversely proportional to the distance.

If we take

$$\phi = \int_A \sigma \ln \frac{1}{r'} da, \tag{283}$$

we have

$$\mathbf{E} = -\nabla\phi, \tag{284}$$

where

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}$$

is the two-dimensional operator. This two-dimensional potential differs from the three-dimensional one in having $\ln \frac{1}{r'}$ instead of $\frac{1}{r}$. For this reason it is called the **logarithmic potential.**

Since

$$\nabla \cdot \mathbf{E} = 4\pi\rho = 2\pi\sigma,$$

from equation (284) we get

$$\nabla^2\phi = -2\pi\sigma. \tag{285}$$

This is analogous to Poisson's equation but with 4π replaced by 2π.

In a similar way, for a surface charge of density $\sigma(\xi, \eta)$ or a double layer of moment $\tau(\xi, \eta)$ on the surface of an infinite cylinder perpendicular to the xy-plane, we obtain the potentials

$$\phi = \int_C \sigma' \ln \frac{1}{r'} ds, \tag{286}$$

$$\phi = \int_C \tau' \mathbf{n} \cdot \nabla \ln \frac{1}{r'} ds, \tag{287}$$

where C is the curve in which the cylinder cuts the xy-plane and σ', τ' are twice the corresponding space densities.

In the analysis of a two-dimensional vector field (whether electric or not) integrals of the types (283), (286), (287) are

obtained. These can be interpreted either as space potentials associated with electric distributions which depend on only two rectangular coördinates, or as plane potentials associated with a type of charge which exerts not the Coulomb force but a force inversely proportional to the distance.

PROBLEMS

1. Show that the solid angle at the vertex of a right circular cone is $2\pi(1 - \cos \alpha)$, where α is the angle between the axis and the generators of the cone.

2. In spherical coördinates show that the solid angle subtended at the origin by the element of surface is

$$d\omega = \sin \theta \, d\theta \, d\phi.$$

3. Find the solid angle subtended at the origin by the part of the plane $y = b$ between $x = -a$ and $x = a$.

4. The curve C forms the complete boundary of an area S. If the z-axis does not intersect the area, show that the solid angle subtended at the origin by S is

$$\int_C -\cos \theta \, d\phi,$$

the integral being taken along C in the direction which is positive with respect to S.

5. Find the solid angle in the preceding problem if the z-axis intersects the surface area, the normal at the point of intersection making an acute angle with the z-axis.

6. If ∇_P and ∇_Q are defined by equations (211), (212) and

$$\phi = f(\xi - x, \eta - y, \zeta - z),$$

show that

$$\nabla_P \phi = -\nabla_Q \phi.$$

7. A sphere of radius b and center at the origin contains a space charge of constant density ρ. By symmetry the electric intensity has the form

$$\mathbf{E}(x, y, z) = f(r)\mathbf{R},$$

where $\mathbf{R}$ is the vector from the origin to $P(x, y, z)$ and r is its length. By applying equation (253) to a sphere S of radius r and center at the origin find $f(r)$ and so $\mathbf{E}$. In this way show that

$$\mathbf{E} = \frac{4\pi\rho b^3 \mathbf{R}}{3r^3},$$

if P lies outside S, and

$$\mathbf{E} = \frac{4\pi\rho \mathbf{R}}{3}$$

if P lies inside S.

8. In the preceding problem show that the potential

$$\phi = \int_P^\infty \mathbf{E} \cdot d\mathbf{R}$$

has the value

$$\phi = \frac{4\pi\rho b^3}{3r}$$

if P lies outside S and

$$\phi = \tfrac{2}{3}\pi\rho(3b^2 - r^2)$$

if inside. Check these expressions by substituting in Poisson's equation.

9. A sphere of radius b contains a space charge of density

$$\rho = b^2 - r^2.$$

By the methods of the last two problems find the values of $\mathbf{E}$ and ϕ outside the sphere and inside. Check by substituting the values of ϕ in Poisson's equation.

10. A cylinder of radius b and axis OZ, extending to infinity in both directions, contains a space charge of constant density ρ. By symmetry the electric intensity has the form

$$\mathbf{E} = f(r)\,\nabla r,$$

where r is the distance from the z-axis. By applying equation (253) to a cylinder S of radius r, altitude unity, and axis OZ, find $f(r)$ and so $\mathbf{E}$, (1) at a point outside the space charge and (2) at a point inside. Show that the integral

$$\int_P^\infty \mathbf{E} \cdot d\mathbf{R}$$

with infinite limit does not converge. By using the integral

$$\phi(P) = \int_P^O \mathbf{E} \cdot d\mathbf{R},$$

where O is the origin, find a potential outside the charge and inside, and show that it satisfies Poisson's equation.

11. A cylinder of radius b and axis OZ contains a space charge of density

$$\rho = r,$$

where r is the distance from the axis. By the method of the preceding problem find the values of $\mathbf{E}$ and ϕ outside the charge and inside, and show that ϕ satisfies Poisson's equation.

12. Upon a sphere of radius b and center at the origin there is a surface charge of constant surface density σ. By a method similar to that used in problem 7 determine $\mathbf{E}$ and ϕ at points outside and inside the sphere. At points of the surface show that the electric intensity satisfies equation (270).

13. A cylinder of radius b and axis OZ, extending both ways to infinity, carries a surface charge of constant density σ. By a method similar to that used in problem 10 find the values of $\mathbf{E}$ and ϕ outside the cylinder and inside, and show that $\mathbf{E}$ satisfies equation (270) at points of the surface.

14. A square plate carries a surface charge of constant density σ. At a point on an edge show that the electric intensity is not finite.

15. Upon the z-axis there is a charge of constant linear density c per unit length. Find the values of E and ϕ in terms of the distance r from the axis.

16. Show that the potential due to a space charge of constant density ρ occupying a region of boundary S is

$$\phi(P) = \frac{\rho}{2}\int_S \frac{\mathbf{n}\cdot\mathbf{R}\,da}{r},$$

where R is the vector and r the distance from P to the element of integration.

17. If P has continuous first derivatives, express the polarization potential

$$\phi = \int_V \mathbf{P}\cdot\nabla\frac{1}{r}\,dv$$

as the difference of the potentials of a surface charge and a space charge.

18. Show that the field due to two doublets of moments $\mathbf{M}_1$, $\mathbf{M}_2$ at the origin is equal to that of a single doublet of moment

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2.$$

19. Express the solid angle subtended by a surface S as the potential of a double layer.

20. Upon a sphere of radius b there is a surface charge of constant density σ. Find a double layer which added to this gives a total potential zero inside the sphere.

21. If the potential due to a certain distribution of charges is zero at all points of a region, show that there are no charges in the region.

CHAPTER VII

HARMONIC FUNCTIONS

66. Introduction. The simplest type of vector field is that of a vector $\mathbf{F}(x, y, z)$ which is irrotational, solenoidal, and continuously differentiable. Since it is irrotational there is a potential ϕ such that

$$\mathbf{F} = -\nabla\phi$$

and since it is solenoidal and continuously differentiable, this potential satisfies Laplace's equation,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0.$$

A solution of Laplace's equation is called a **harmonic function.** The function is called harmonic at a point if there is a neighborhood* about the point in which its second derivatives exist, are continuous, and satisfy Laplace's equation. It is called harmonic in a region if it is harmonic at interior points and continuous in the region.

A vector field is usually regarded as having its origin in places where the divergence and curl are different from zero. Harmonic functions are thus usually associated with regions free from material which gives rise to the field. For example, the potential in an electrostatic field is harmonic in regions not containing charges or dielectrics. The potential in a steady magnetic field is harmonic in regions free from currents and magnetized material. In a fluid of constant density the velocity potential is harmonic in regions where there are no sources or eddies. The temperature of a medium in a steady state is harmonic in regions where heat is not produced or absorbed. The potential in a gravitational field is harmonic in regions which do not contain matter.

* A neighborhood of a point P is a small region which includes all points within some distance δ from P.

67. Laplace's Equation in Orthogonal Coördinates. It is often convenient to have Laplace's equation expressed in terms of general coördinates u_1, u_2, u_3. To obtain the appropriate expressions we consider $\nabla^2\phi$ as the divergence of the gradient of ϕ.

From (162) the gradient of ϕ is

$$\nabla\phi = \frac{\mathbf{i}_1}{h_1}\frac{\partial\phi}{\partial u_1} + \frac{\mathbf{i}_2}{h_2}\frac{\partial\phi}{\partial u_2} + \frac{\mathbf{i}_3}{h_3}\frac{\partial\phi}{\partial u_3},$$

where $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are unit vectors tangent to the coördinate curves and h_1, h_2, h_3 (equation (142)) are the derivatives of arc along those curves. The gradient is thus a vector of components

$$\frac{1}{h_1}\frac{\partial\phi}{\partial u_1}, \qquad \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}, \qquad \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}$$

along $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ respectively. Substituting these values for f_1, f_2, f_3 in (166) we obtain

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)\right]. \quad (288)$$

Equated to zero this gives Laplace's equation.

To obtain Laplace's equation in a particular set of orthogonal coördinates we thus merely determine the arc derivatives h_1, h_2, h_3, substitute their values in (288), and equate the result to zero. Since each of the h's appears twice as a factor in (288) the algebraic signs of these quantities are immaterial.

In cylindrical coördinates r, θ, z, for example,

$$h_1 = 1, \quad h_2 = r, \quad h_3 = 1.$$

Laplace's equation is therefore

$$\nabla^2\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial\phi}{\partial z}\right)\right] = 0,$$

which is equivalent to

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0.$$

68. Boundary Value Relations. If a function $\phi(x, y, z)$ is continuous and differentiable at a point Q on the boundary of a region, we use the notation

$$\frac{\partial\phi}{\partial n} = \mathbf{n}\cdot\nabla\phi$$

to represent the derivative of ϕ in the direction of the outward drawn normal. This is evidently the negative of the derivative in the direction of the inward drawn normal. In many cases values of ϕ at points outside the region are not defined. In such cases, if ϕ has a derivative in the direction of the inward drawn normal, $\frac{\partial \phi}{\partial n}$ is therefore defined as the negative of this derivative. In either case $\frac{\partial \phi}{\partial n}$ is called the **normal derivative** of ϕ. At a point of the boundary where ϕ is discontinuous neither of these definitions may be applicable. But if ϕ is continuously differentiable in a closed region and $\mathbf{n}$ is the normal at a point Q on the boundary,

$$\frac{\partial \phi(P)}{\partial n}$$

taken in the direction of $\mathbf{n}$ at an interior point P will approach a limit when P approaches Q. This limit defines the normal derivative at Q.

If ϕ_1 and ϕ_2 are harmonic and continuously differentiable in a closed region V of boundary S,

$$\int_S \left(\phi_2 \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial \phi_2}{\partial n} \right) da = 0. \tag{289}$$

To prove this let

$$\mathbf{F} = \phi_2 \nabla \phi_1 - \phi_1 \nabla \phi_2$$

and let V' be a region of boundary S' within S. At each point of this region and its boundary $\mathbf{F}$ has continuous derivatives which satisfy the equation

$$\nabla \cdot \mathbf{F} = \phi_2 \nabla^2 \phi_1 - \phi_1 \nabla^2 \phi_2 = 0.$$

The divergence theorem therefore gives

$$\int_{S'} \mathbf{F} \cdot \mathbf{n} \, da = \int_{S'} \mathbf{n} \cdot (\phi_2 \nabla \phi_1 - \phi_1 \nabla \phi_2) da = 0.$$

Suppose now S' approaches S as limit. Since $\mathbf{F}$ is continuous in the closed region V, the last equation gives (289) as limit.

As a particular case let

$$\phi_1 = \phi$$

be harmonic and continuously differentiable in a closed region of boundary S and let

$$\phi_2 = \frac{1}{r},$$

where r is the distance from the point $P(x, y, z)$. If P lies inside the region let S' be a sphere of center P and radius r' less than the shortest distance from P to S. In the region between S' and S the functions ϕ_1, ϕ_2 are both harmonic. Applied to this region equation (289) gives

$$\int_S \left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]da + \int_{S'}\left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]da = 0. \quad (290)$$

In the second integral the normal is directed outward from the region of integration, that is, toward P. Since r has the constant value r' on S' and ϕ is harmonic in S

$$\int_{S'}\frac{1}{r}\frac{\partial\phi}{\partial n}da = \frac{1}{r'}\int_{S'}\mathbf{n}\cdot\nabla\phi\, da = \frac{1}{r'}\int\nabla^2\phi\, dv = 0.$$

Also

$$-\int_{S'}\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)da = -\int_{S'}\phi\, d\omega,$$

where $d\omega$ is the solid angle subtended by da at P. As r' is made smaller and smaller this approaches the limit

$$-4\pi\phi(P).$$

From (290) we thus get

$$\phi(P) = \frac{1}{4\pi}\int_S\left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]da.$$

But if P lies outside S the functions

$$\phi_1 = \phi, \qquad \phi_2 = \frac{1}{r}$$

are both harmonic inside and so (289) gives

$$\int_S\left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]da = 0.$$

We have thus proved the following theorem:

If ϕ is harmonic and continuously differentiable in a closed region of boundary S and r is the distance from the point $P(x, y, z)$,

$$\phi(P) = \frac{1}{4\pi}\int_S \left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]da, \tag{291}$$

if P lies inside the region and

$$0 = \int_S \left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]da, \tag{292}$$

if P lies outside.

Equations (289), (291), (292) are special cases of formulas first given by George Green.* Their remarkable feature is that the integrals contain only values of the functions and their derivatives at points of the boundary.

69. Analytic Properties. The quantity r in (291) is the distance between $P(x, y, z)$ and the point $Q(\xi, \eta, \zeta)$ at which the element of area da is located. If P lies within S the integrand is then a function of x, y, z with derivatives of all orders uniformly continuous for all positions of Q on S. Therefore $\phi(P)$ is a function of x, y, z which at points within S has continuous derivatives of all orders.

In these equations P may be any point at which ϕ is harmonic and S a sphere containing P and belonging to the neighborhood about P in which ϕ is harmonic. *If a function is harmonic at a point all of its partial derivatives therefore exist and are continuous at that point.*

From the equation

$$\nabla^2\phi = 0$$

by partial differentiation we obtain

$$\frac{\partial}{\partial x}\nabla^2\phi = \nabla^2\frac{\partial\phi}{\partial x} = 0.$$

If ϕ is a solution of Laplace's equation, $\dfrac{\partial\phi}{\partial x}$ is thus also a solution. In a similar way we show that *if ϕ is harmonic at a point all of its partial derivatives are also harmonic at that point.*

To investigate the behavior of a harmonic function near a

* George Green, "Essay on the Application of Mathematical Analysis to the Theory of Electricity and Magnetism," 1828.

point O, take O as origin and let S be a sphere of center O and radius r_0 lying within a region in which ϕ is harmonic. Using

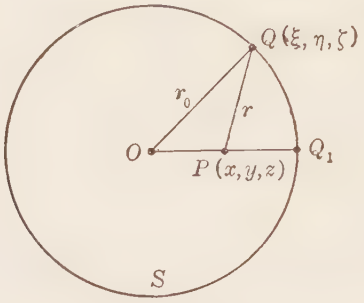


FIG. 59

the letters $\mathbf{P}$, $\mathbf{Q}$ to represent the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and writing r_0^2 instead of $\mathbf{Q}^2$, we have

$$r^2 = (\mathbf{Q} - \mathbf{P})^2 = r_0^2 - 2\mathbf{P} \cdot \mathbf{Q} + \mathbf{P}^2.$$

Hence

$$\frac{1}{r} = [r_0^2 - (2\mathbf{P} \cdot \mathbf{Q} - \mathbf{P}^2)]^{-\frac{1}{2}}. \tag{293}$$

For small values of $\mathbf{P}$

$$2\mathbf{P} \cdot \mathbf{Q} - \mathbf{P}^2$$

is less in absolute value than r_0^2 and so (293) can be expanded by the binomial theorem in powers of this quantity. Writing

$$u = 2\mathbf{P} \cdot \mathbf{Q} = 2(\xi x + \eta y + \zeta z),$$
$$-v = \mathbf{P}^2 = x^2 + y^2 + z^2,$$

the resulting series has the form

$$\sum a_n (u + v)^n. \tag{294}$$

We can expand the powers of $u + v$, collect terms of the same degree in x, y, z and so get

$$\frac{1}{r} = H_0 + H_1(x, y, z) + \cdots + H_n(x, y, z) + \cdots \tag{295}$$

where $H_n(x, y, z)$ is a homogeneous polynomial of nth degree in x, y, z.

To justify the rearrangement by which we pass from (294) to (295) it should be shown that (294) converges when u, v, a_n are replaced by their absolute values.* From the binomial theorem the a's are all positive. If Q_1 is the point in which OP cuts S,

$$u = 2\mathbf{P} \cdot \mathbf{Q} \leqq 2\mathbf{P} \cdot \mathbf{Q}_1.$$

If we take

$$u_1 = 2\mathbf{P} \cdot \mathbf{Q}_1, \qquad v_1 = \mathbf{P}^2,$$

the quantity

$$[r_0^2 - (2\mathbf{P} \cdot \mathbf{Q}_1 + \mathbf{P}^2)]^{-\frac{1}{2}}$$

has a binomial expansion of the form

$$\sum a_n(u_1 + v_1)^n \tag{296}$$

and this converges when

$$2\mathbf{P} \cdot \mathbf{Q}_1 + \mathbf{P}^2 < r_0^2.$$

Since

$$2\mathbf{P} \cdot \mathbf{Q}_1 = 2r_0 |\mathbf{P}|,$$

this inequality evidently holds if

$$|\mathbf{P}| \leqq \tfrac{1}{3} r_0.$$

For points P satisfying this condition the terms of (294) are then not greater than the corresponding terms of the absolutely convergent series (296). The rearrangement is therefore justified and at the same time it is shown that (295) converges uniformly for all positions of Q on S.

The expression (295) can therefore be substituted for $\frac{1}{r}$ in (291) and the result integrated over S, giving

$$\phi(x, y, z) = \phi_0 + \phi_1(x, y, z) + \cdots + \phi_n(x, y, z) + \cdots \tag{297}$$

where $\phi_n(x, y, z)$ is a homogeneous polynomial of nth degree in x, y, z. *About a point O where ϕ is harmonic there is then a region in which ϕ can be represented by a convergent power series.* A function having this property is called **analytic.**

* cf. F. S. Woods, "Advanced Calculus," page 59.

The power series (297) can be determined as the Maclaurin expansion in the neighborhood of O. Except for certain factorial divisors the coefficients in this series are thus values of the partial derivatives of ϕ at O. In particular, if ϕ is constant in a small region about O, these are all zero and ϕ is constant throughout the range in which the series is valid. Thus a function, harmonic in a connected three-dimensional region* and constant in a neighborhood however small about one of its points, is constant throughout the region.

70. Average Value Theorem. *A function harmonic in a closed sphere has at the center a value equal to the average of its values on the surface.*

By this we mean that if ϕ is harmonic at interior points of a sphere and is continuous in the closed region consisting of these points and the surface of the sphere,

$$\phi(P) = \frac{\int_S \phi \, da}{\int_S da} = \frac{1}{4\pi r^2}\int_S \phi \, da, \tag{298}$$

where P is the center of the sphere, r its radius and S its surface. To prove this let S' be a sphere of radius $r' < r$. In this sphere ϕ will be harmonic and continuously differentiable. Consequently (equation (291))

$$\phi(P) = \frac{1}{4\pi}\int_{S'} \frac{\mathbf{n}\cdot\nabla\phi}{r} \, da - \frac{1}{4\pi}\int_{S'} \phi \mathbf{n}\cdot\nabla\frac{1}{r} \, da. \tag{299}$$

Since r is constant on S' the first integral on the right has the value

$$\frac{1}{4\pi r'}\int_{S'} \mathbf{n}\cdot\nabla\phi \, da = \frac{1}{4\pi r'}\int \nabla^2\phi \, dv = 0.$$

Also

$$-\mathbf{n}\cdot\nabla\frac{1}{r} = \frac{1}{r'^2}.$$

Consequently (299) is equivalent to

$$\phi(P) = \frac{1}{4\pi r'^2}\int_{S'} \phi \, da.$$

* The region is assumed to be such that any two interior points can be connected by a polygonal line all points of which are interior.

Since ϕ is continuous in the closed sphere S, this gives (298) as limit when r' approaches r.

From (298) we have

$$4\pi r^2\phi(P) = \int_S \phi \, da.$$

Multiplying by dr and integrating between the limits zero and r_1, we get

$$\tfrac{4}{3}\pi r_1^3\phi(P) = \int_{V_1} \phi \, dv,$$

where V_1 is the sphere of center P and radius r_1. Hence

$$\phi(P) = \frac{1}{\tfrac{4}{3}\pi r_1^3}\int_{V_1} \phi \, dv.$$

This expresses that *a function harmonic in a closed sphere has at the center a value equal to the average of its values in the sphere.*

71. Infinite Regions. The regions hitherto considered have all been finite. A function is called harmonic in the infinite region outside a closed surface if it is harmonic in each finite sub-region and has the limit zero at infinity. In connection with such functions the following theorem is of fundamental importance.

A function harmonic at every finite point and having a limit at infinity is constant.

By saying that ϕ has a limit c at infinity we mean that to each positive number ϵ there corresponds a sphere S outside which the condition

$$|\phi - c| < \epsilon \tag{300}$$

is everywhere satisfied. Suppose ϕ satisfies this condition and is harmonic at every finite point. If P is any point we can find a sphere S' of center P which encloses S. On this sphere $\phi - c$ is harmonic and everywhere in numerical value less than ϵ. Its average value (equation (298)) $\phi(P) - c$ must satisfy the same condition. Hence

$$|\phi(P) - c| < \epsilon.$$

Since ϵ may be taken as small as we please, $\phi(P)$ must then have the constant value c.

Most theorems concerning functions harmonic in a finite region are also true for functions harmonic in the infinite region outside a closed surface. Since the properties of harmonic functions are usually deduced from the boundary value relations (289), (291), (292), the simplest way to prove this is to show that those boundary relations are also satisfied by functions harmonic in the outside region.

First we consider (291). Suppose ϕ continuously differentiable at points of a closed surface S and harmonic in the region V outside S. If P is any point in this region, we can find a sphere S' containing both S and P. In the region V' between S and S' equation (291) is satisfied but with the surface integral extended over the whole boundary $(S + S')$ of V'. If we write

$$\psi = \frac{1}{4\pi}\int_S \left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]da, \tag{301}$$

$$\psi' = \frac{1}{4\pi}\int_{S'} \left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]da, \tag{302}$$

the normal in each case extending outward from V', we then have

$$\phi(P) = \psi(P) + \psi'(P). \tag{303}$$

Since $\phi(P)$ and $\psi(P)$ depend only on S and P, and have the limit zero at infinity, the function $\psi'(P)$ is independent of S' and also has the limit zero at infinity. Furthermore $\psi'(P)$ is harmonic at all points P inside S'. Since S' may be as large as we please $\psi'(P)$ is then harmonic at all finite points and has the limit zero at infinity. It is therefore zero and $\phi = \psi$, which proves (291). If P lies inside S, by a similar argument we obtain $\psi = 0$, which proves (292).

In these equations it should be noticed, however, that the normal at points of S is directed outward from the region V, that is, toward what is ordinarily called the inside of S.

Let the origin be a point inside the surface S and let

$$r_0 = \sqrt{x^2 + y^2 + z^2}$$

be its distance from $P(x, y, z)$. Equation (291) gives

$$r_0\phi(P) = \frac{1}{4\pi}\int_S \left[\frac{r_0}{r}\frac{\partial\phi}{\partial n} - r_0\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]da,$$

$$r_0^2\nabla\phi(P) = \frac{1}{4\pi}\int_S \left[r_0^2\nabla_P\frac{1}{r}\frac{\partial\phi}{\partial n} - r_0^2\phi\frac{\partial}{\partial n}\left(\nabla_P\frac{1}{r}\right)\right]da.$$

Since S is finite, when r_0 is sufficiently large the ratio

$$\frac{r_0}{r}$$

will be approximately unity at all points of S. When P moves away indefinitely (r_0 tending to infinity) these equations therefore show that ϕ and $\nabla\phi$ tend to zero in such a way that $r_0\phi$ and $r_0^2\nabla\phi$ remain bounded. A function having this property is sometimes called **regular at infinity.** *A function harmonic outside a closed surface is thus regular at infinity.*

We are now in position to prove the remaining one of the boundary relations, namely (289), for the outside region. To do this suppose ϕ_1, ϕ_2 are harmonic and continuously differentiable in the region consisting of a closed surface S and points outside. Let S' be a large sphere with center at the origin and containing S. Applied to the region between S and S' equation (289) gives

$$\int_S \left(\phi_2 \frac{\partial\phi_1}{\partial n} - \phi_1 \frac{\partial\phi_2}{\partial n}\right) da + \int_{S'} \left(\phi_2 \frac{\partial\phi_1}{\partial n} - \phi_1 \frac{\partial\phi_2}{\partial n}\right) da = 0. \quad (304)$$

Since ϕ_1 and ϕ_2 are regular at infinity, there is a number m such that for sufficiently large values of r_0,

$$r_0^3 |\phi_2 \nabla\phi_1 - \phi_1 \nabla\phi_2| < m.$$

The second integral in (304) is therefore in numerical value less than

$$\int_{S'} \frac{m}{r_0^3} da = \frac{4\pi m}{r_0}.$$

When r_0 tends to infinity (304) thus gives (289) as limit.

In hydrodynamics it is usually not the potential ϕ but the velocity $-\nabla\phi$ which is directly known. In such cases the following theorem is important.

If ϕ is harmonic outside a closed surface and $\nabla\phi$ has the limit zero at infinity, ϕ has a limit at infinity.

To show this let P be any point and S' a sphere enclosing S and P. As in (303) we divide ϕ into two parts,

$$\phi = \psi + \psi', \quad (305)$$

representing integrals over S and S' respectively. Then

$$\nabla\phi = \nabla\psi + \nabla\psi'.$$

By hypothesis $\nabla\phi$ has the limit zero at infinity. From (301) it is clear that $\nabla\psi$ has the same property. Hence $\nabla\psi'$ has the limit zero at infinity. The components of $\nabla\psi'$ are then harmonic everywhere with limits zero at infinity. These components are therefore zero and so ψ' is constant. From (305) it follows that ϕ has a limit (namely ψ') at infinity.

72. Maximum and Minimum Properties. *Unless it is constant a function cannot attain its greatest or least value within a region where it is harmonic.*

In fact suppose ϕ is harmonic in a region V and has at an interior point a value m which is not exceeded at any other point of V. Consider the totality of points at which ϕ has this same value m. If these do not fill the region V, they will have one or more frontier points within V. If P_1 is such a point there will be a sphere of center P_1 belonging to V upon which ϕ has values different from m. But this is impossible. For equation (298) applied to this sphere gives

$$\int (m - \phi) da = 0$$

and the integral of a continuous function which is sometimes positive and never negative cannot be zero. Hence ϕ has the value m at all points of V.

In a similar way we show that ϕ is constant if it attains its least value at an interior point.

If it is not constant the magnitude of $\nabla\phi$ cannot attain its greatest value within a region where ϕ is harmonic.

To show this take the x-axis parallel to the gradient at an interior point P. At this point $\nabla\phi$ then has the same magnitude as

$$\frac{\partial\phi}{\partial x}.$$

This function, being harmonic, cannot have a maximum value at P. Consequently there are points as near as we please to P where $\left(\frac{\partial\phi}{\partial x}\right)^2$, and so also

$$|\nabla\phi| = \sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2}$$

has values greater than at P.

The gradient of ϕ can however have a minimum magnitude, for example zero, within a region where ϕ is harmonic.

73. Boundary Value Properties. *A function harmonic in a finite closed region and constant on its boundary is constant in the region.*

For otherwise it would attain its greatest or least value at some point inside the region.

A function may however be harmonic in the infinite region outside a surface S and constant on S but not constant in the whole region. For instance, if r is the distance from P and S is the sphere of center P, the function $\frac{1}{r}$ is constant on S and harmonic but not constant outside S.

Two functions harmonic in a closed region and having equal values on its boundary are equal throughout the region.

For their difference, being harmonic in the region and zero on its boundary, is zero at all points of the region.

This theorem is valid for the infinite region outside a surface as well as for a finite region. In either case a harmonic function is thus uniquely determined by its boundary values.

If a function is harmonic and continuously differentiable in a closed region and its normal derivative is zero at all points of the boundary, it is constant in the region.

Suppose as a first case that ϕ satisfies these conditions in a finite region V of boundary S. Since

$$\nabla \cdot (\phi \nabla\phi) = \phi \nabla^2\phi + (\nabla\phi)^2 = (\nabla\phi)^2,$$

the divergence theorem gives

$$\int_S \phi \frac{\partial\phi}{\partial n} da = \int_V (\nabla\phi)^2 dv. \tag{306}$$

If the normal derivative is zero at all points of S,

$$\int_V (\nabla\phi)^2 dv = 0. \tag{307}$$

Since the integrand is never negative, $\nabla\phi$ is zero and so ϕ is constant at all points of V.

If V is the infinite region outside a surface S, let S' be a sphere of fixed center enclosing S. Applied to the finite region V'

between S and S', equation (306) gives

$$\int_S \phi \frac{\partial \phi}{\partial n} da + \int_{S'} \phi \frac{\partial \phi}{\partial n} da = \int_{V'} (\nabla \phi)^2 \, dv.$$

As the radius r' of S' tends to infinity,

$$r'^2 \phi \frac{\partial \phi}{\partial n}$$

tends to zero. Since $da = r'^2 \, d\omega$, the integral over S' therefore approaches zero and in the limit we again get equation (307). From this we show as before that ϕ is constant, and since ϕ is by definition zero at infinity, the constant is zero.

In a finite region a harmonic function is thus determined except for an added constant by the values of its normal derivative on the boundary, but in the infinite region outside it is uniquely determined.

74. Green's Function. We have seen that a function ϕ harmonic in a region V is determined by its values on the boundary. One might expect a formula would exist expressing ϕ in terms of boundary values. By use of the following physically defined function, Green obtained such a formula.

At an interior point P of the region place a unit positive charge of electricity and let the surface S be a grounded conductor. Charges will be induced on the conductor and will so distribute themselves that the total potential at points of S is zero. At any other point Q of the region these induced charges determine a potential $\psi(Q, P)$. The total potential at Q due to the unit charge at P and the induced charges is

$$G(Q, P) = \frac{1}{r} + \psi(Q, P), \tag{308}$$

where r is the distance between P and Q.

This quantity $G(Q, P)$ is known as **Green's function.** It is a function of P and Q and also depends on the region. In a given closed region it is mathematically defined by the following properties:

(1) It is a function of the form

$$G(Q, P) = \frac{1}{r} + \psi(Q, P),$$

where r is the distance PQ and $\psi(Q, P)$ is harmonic in the coördinates of Q.

(2) It is zero if Q is a point on the boundary.

That these properties uniquely determine the function follows from the fact that if two functions agree in these respects their difference, considered as a function of Q, is harmonic in the closed region and zero on its boundary.

If ϕ is harmonic and continuously differentiable in a closed region of boundary S and P is an interior point,

$$\phi(P) = -\frac{1}{4\pi}\int_S \phi(Q)\,\frac{\partial G(Q, P)}{\partial n}\,da, \tag{309}$$

where $G(Q, P)$ is the Green's function for the region.

This is Green's formula expressing the value of a harmonic function at an interior point in terms of values assumed by the same function at points on the boundary. To prove the formula we note first that since ϕ and ψ are both harmonic and continuously differentiable, by equations (289) and (291)

$$0 = \frac{1}{4\pi}\int_S \left[\psi(Q, P)\,\frac{\partial \phi(Q)}{\partial n} - \phi(Q)\,\frac{\partial \psi(Q, P)}{\partial n}\right] da,$$

$$\phi(P) = \frac{1}{4\pi}\int_S \left[\frac{1}{r}\,\frac{\partial \phi(Q)}{\partial n} - \phi(Q)\,\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] da.$$

Adding these equations and using the relation

$$G(Q, P) = \frac{1}{r} + \psi(Q, P) = 0,$$

satisfied for points Q on the boundary, we obtain (309).

The Green's function for a region satisfies the condition

$$G(Q, P) \geqq 0 \tag{310}$$

for all positions of P and Q in the region. Certainly this is true when Q is near P because $\frac{1}{r}$ in (308) is then very large. Keeping P fixed, if $G(Q, P)$ were negative for some points Q in the region, there would be at least one Q_0 for which this function has a minimum value. But this is impossible because Q_0 cannot be a boundary point, since $G(Q, P)$ would then be zero, and cannot be an interior point, since $G(Q, P)$ would there be harmonic in Q.

It follows that when Q is on the boundary

$$\frac{\partial G(Q, P)}{\partial n} \leqq 0. \tag{311}$$

For $G(Q, P)$, considered as a function of Q, being zero on the boundary and positive inside, has in the direction of the inward drawn normal a derivative which is zero or positive, and so in the opposite direction one which is zero or negative.

Finally, if P and Q are both interior points

$$G(Q, P) = G(P, Q). \tag{312}$$

To show this let P_1, P_2 be interior points of the region and S_1, S_2 small spheres with these points as centers. In the region bounded by S_1, S_2, S the functions $G(Q, P_1)$, $G(Q, P_2)$ are both harmonic in the coördinates of Q. From equation (289) the integral of the function

$$f(Q) = G(Q, P_1)\frac{\partial G(Q, P_2)}{\partial n} - G(Q, P_2)\frac{\partial G(Q, P_1)}{\partial n}$$

over the total boundary of the region is zero. Since

$$G(Q, P_1) = G(Q, P_2) = 0$$

when Q is on S, we then have

$$\int_{S_1} f\, da + \int_{S_2} f\, da = 0. \tag{313}$$

Since $G(Q, P_2)$ is harmonic in the sphere S_1 and the normal is directed inward, from (289) and (291) we have

$$\int_{S_1}\left[\psi(Q, P_1)\frac{\partial G(Q, P_2)}{\partial n} - G(Q, P_2)\frac{\partial \psi(Q, P_1)}{\partial n}\right] da = 0,$$

$$\int_{S_1}\left[\frac{1}{r_1}\frac{\partial G(Q, P_2)}{\partial n} - G(Q, P_2)\frac{\partial}{\partial n}\left(\frac{1}{r_1}\right)\right] da = -4\pi G(P_1, P_2),$$

where r_1 is the distance from P_1. By adding these equations we obtain

$$\int_{S_1} f\, da = -4\pi G(P_1, P_2).$$

In a similar way we prove

$$\int_{S_2} f\, da = 4\pi G(P_2, P_1).$$

Equation (313) is therefore equivalent to

$$G(P_1, P_2) = G(P_2, P_1),$$

which was to be proved.

The Green's function is thus symmetric with respect to P and Q. It may therefore be considered either as the potential at Q due to a unit charge at P with boundary potential kept zero, or as the potential at P due to a unit charge at Q and boundary potential zero. It satisfies Laplace's equation in the coördinates of either point and is zero if either point is on the boundary.

75. The Dirichlet Problem. A very famous problem is concerned with the question whether there exists a function $\phi(x, y, z)$ harmonic in a given closed region V and having upon its boundary the same values as a given continuous function $f(x, y, z)$. A physical reason for believing such a function exists is that if the value $f(x, y, z)$ is given to the temperature at each point $Q(x, y, z)$ on the boundary, then, after thermal equilibrium is established, the temperature in the interior will be represented by a function which has the required properties.

Green solved the problem by using his physically defined function $G(Q, P)$. Under conditions normally satisfied he showed that if $\phi(Q)$ in equation (309) is replaced by $f(Q)$ the resulting integral

$$-\frac{1}{4\pi}\int_S f(Q)\,\frac{\partial G(Q, P)}{\partial n}\, da \tag{314}$$

represents a function which is harmonic at each interior point P of the region and which approaches $f(Q)$ as limit when P approaches a point Q on the boundary.

Dirichlet first endeavored to prove the existence of such a function by purely mathematical methods. As a consequence it is usually called the **Dirichlet problem**, or first boundary value problem of potential theory. For very general types of surfaces, including all ordinary ones, the existence of the function has been proved. But it has been shown that surfaces exist for which there is no solution.

To solve the problem for a particular region we can either proceed at once to find the function $\phi(x, y, z)$ with given boundary values $f(x, y, z)$, or we can first demonstrate mathematically that in the given region there exists a function $G(Q, P)$ having the properties assumed by Green and then use (314) to determine ϕ.

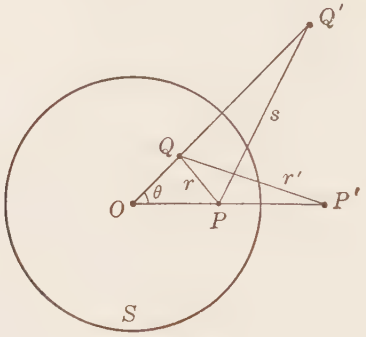


Fig. 60

For our purposes it will be sufficient to consider only the case of the sphere and we shall use the second method.

Let S be a sphere of center O and radius b. With each point P associate its inverse with respect to the sphere. This is defined as the point P' on the line OP such that the product of the distances OP, OP' is equal to the square of the radius. Similarly, let Q and Q' be inverse points. For brevity denote the distances OP, OP', OQ, OQ', QP, QP', PQ' by p, p', q, q', r, r', s respectively. Then

$$pp' = qq' = b^2, \tag{315}$$

whence

$$\frac{p}{q'} = \frac{q}{p'}.$$

This shows that the triangles OPQ' and OQP' with common angle θ at O are similar. Consequently

$$pr' = qs. \tag{316}$$

The quantity

$$G(Q, P) = \frac{1}{r} - \frac{b}{pr'} = \frac{1}{r} - \frac{b}{qs} \tag{317}$$

is Green's function for the sphere. To prove this we should show that

$$\psi(Q, P) = -\frac{b}{pr'},$$

is harmonic in the coördinates of Q and that $G(Q, P)$ is zero when Q is on the boundary. That ψ is harmonic follows since it is the electrostatic potential due to a charge $-\frac{b}{p}$ at P'. That $G(Q, P) = 0$ when Q is on the boundary follows since Q' then coincides with Q, $q = b$, $r = s$, and so

$$G(Q, P) = \frac{1}{r} - \frac{b}{qs} = 0.$$

From the general theory (§74) or from equation (317) it is also clear that

$$G(Q, P) = G(P, Q)$$

and that this function is harmonic in the coördinates of P.

Next let Q be taken on the sphere at the element of integration and

$$\phi(P) = -\frac{1}{4\pi}\int_S f(Q)\frac{\partial G(Q, P)}{\partial n}\,da, \tag{318}$$

where $f(Q)$ is continuous on S. We shall show that $\phi(P)$ is harmonic in the sphere and approaches $f(Q)$ as limit when P approaches a point Q on the surface.

That $\phi(P)$ is harmonic within the sphere follows since $G(Q, P)$ considered as a function of P is harmonic and continuously differentiable. Since the Green's function for the sphere has the properties assumed in proving (309) and a constant is the simplest kind of harmonic function, we can take $\phi = 1$ in equation (309) and so obtain

$$1 = -\frac{1}{4\pi}\int_S \frac{\partial G(Q, P)}{\partial n}\,da. \tag{319}$$

If Q_0 is any fixed point on the sphere we then have

$$f(Q_0) = -\frac{1}{4\pi}\int_S f(Q_0)\frac{\partial G(Q, P)}{\partial n}\,da,$$

whence by subtraction

$$\phi(P) - f(Q_0) = -\frac{1}{4\pi}\int_S [f(Q) - f(Q_0)]\frac{\partial G(Q, P)}{\partial n}\, da. \quad (320)$$

Let S_0 be a small region about Q_0 within which

$$|f(Q) - f(Q_0)| < \epsilon \quad (321)$$

and let S_1 be the remainder of S. The quantity $G(Q, Q_0)$ is zero for all positions of Q except $Q = Q_0$. The derivatives of this quantity with respect to the coördinates of Q are therefore zero. Also, if Q is not too near P, $\dfrac{\partial G(Q, P)}{\partial n}$ is a continuous function of P. When P approaches Q_0 the part of (320) taken over S_1 therefore approaches zero. Equations (311), (319), (321) show that the part taken over S_0 is in absolute value less than

$$\frac{1}{4\pi}\int_{S_0}\epsilon\left|\frac{\partial G(Q, P)}{\partial n}\, da\right| \leqq \frac{1}{4\pi}\int_S \epsilon\left|\frac{\partial G(Q, P)}{\partial n}\, da\right| = \epsilon.$$

Hence $\phi(P) - f(Q_0)$ approaches zero and so $\phi(P)$ approaches $f(Q_0)$ as limit.

Thus (318) is the solution of the Dirichlet problem for the region inside the sphere if $G(Q, P)$ has the value

$$G(Q, P) = \frac{1}{r} - \frac{b}{pr'}.$$

In a similar way we show that if P lies outside the sphere, this is the Green's function for the outside region and (318) is the solution of the Dirichlet problem for that region. It should be noted however that the normal in the latter case extends outward from the region, that is, toward the inside of the sphere.

76. Poisson's Integral. Before the appearance of Green's work an equation equivalent to (309) was given by Poisson for the sphere. To obtain this we use the notation of Fig. 60. From the diagram and equation (315) it is clear that

$$r^2 = p^2 + q^2 - 2pq\cos\theta,$$

$$r'^2 = \frac{b^4}{p^2} + q^2 - 2\frac{b^2}{p}q\cos\theta.$$

Considering r and r' as functions of p, q, θ, we have

$$\frac{\partial}{\partial q}\left(\frac{1}{r}\right) = \frac{p\cos\theta - q}{r^3},$$

$$\frac{\partial}{\partial q}\left(\frac{1}{r'}\right) = \frac{b^2\cos\theta - pq}{pr'^3}.$$

Multiplying the second equation by $\frac{b}{p}$, subtracting from the first, and using the relation

$$pr' = br,$$

which is valid when Q is on the sphere, we get

$$\frac{\partial}{\partial q}\left(\frac{1}{r} - \frac{b}{pr'}\right) = \frac{p^2 - b^2}{br^3}.$$

At a point of the sphere the partial derivative with respect to q is the derivative in the direction of the outward drawn normal. The last equation is therefore equivalent to

$$\frac{\partial}{\partial n}G(Q, P) = \frac{p^2 - b^2}{br^3},$$

and equation (309) becomes

$$\phi(P) = \frac{b^2 - p^2}{4\pi b}\int_S \frac{\phi(Q)\,da}{r^3}. \tag{322}$$

This is known as **Poisson's integral.** In deriving it we have assumed that ϕ is continuously differentiable. By considering first the case of an inner sphere and then allowing this to approach S as limit, we can prove its validity for any function harmonic in the closed sphere S. By replacing $\phi(Q)$ by $f(Q)$ we obtain Poisson's solution of the Dirichlet problem for the sphere.

Suppose ϕ is everywhere harmonic and positive. We then have

$$\int_S \frac{\phi(Q)\,da}{r^3} = \frac{1}{\bar{r}^3}\int_S \phi(Q)\,da,$$

where $\bar{r}$ is a mean value of r, that is, a value between the least and the greatest distance from P to the sphere. Also by the

average value theorem

$$\int_S \phi(Q)\, da = 4\pi b^2 \phi(O),$$

where $\phi(O)$ is the value of ϕ at the center of the sphere. Hence (322) is equivalent to

$$\phi(P) = \frac{(b^2 - p^2)b}{\bar{r}^3}\phi(O). \tag{323}$$

Keeping O and P fixed, let the radius b increase indefinitely. The quantity

$$\left(\frac{b^3}{r^3} - \frac{p^2 b}{r^3}\right),$$

and so also its mean value, approaches unity for all positions of Q on the sphere. Equation (323) therefore gives as limit

$$\phi(P) = \phi(O).$$

Since P may be any point we have thus proved that a function is constant if it is everywhere harmonic and positive.

More generally, *a function is constant if it is everywhere harmonic and has either an upper or a lower bound.* If, for example, it has an upper bound, there is a number c such that ϕ is everywhere less than c. If ϕ is everywhere harmonic, the function $c - \phi$ (and consequently ϕ) is constant since it is everywhere harmonic and positive.

A function harmonic everywhere is thus either constant or else takes every finite value.

77. The Neumann Problem. We have seen that except for an added constant a function harmonic in a region is determined by the values of its normal derivative at points of the boundary. The question arises whether there exists a function ϕ harmonic in a given region and having upon its boundary S a normal derivative equal to a given continuous function $f(x, y, z)$. This is known as **Neumann's problem**, or the second boundary value problem of potential theory. If the region is finite and such a function exists the divergence theorem requires

$$\int_S f\, da = \int_S \frac{\partial \phi}{\partial n}\, da = \int_V \nabla^2 \phi\, dv = 0. \tag{324}$$

For very general regions, including all ordinary ones, it has been shown that when this condition is satisfied the function does exist.

One way to solve the problem is to find a function

$$H(Q, P) = \frac{1}{r} + \chi(Q, P) \tag{325}$$

where χ (Q, P) is harmonic in the coördinates of Q and for points Q on the boundary satisfies the condition

$$\frac{\partial H(Q, P)}{\partial n} = c, \tag{326}$$

where c is constant. Analogous to (312), this function H (Q, P) is usually assumed to be symmetric in the coördinates of P and Q,

$$H(Q, P) = H(P, Q), \tag{327}$$

and so is also harmonic in the coördinates of P. If $\phi(P)$ is any function harmonic and continuously differentiable in the given region (equations (289) and (291))

$$\phi(P) = \frac{1}{4\pi}\int_S \left[\frac{1}{r}\frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] da,$$

$$0 = \frac{1}{4\pi}\int_S \left[\chi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \chi}{\partial n}\right] da,$$

whence by addition

$$\phi(P) = \frac{1}{4\pi}\int_S H(Q, P)\frac{\partial \phi}{\partial n}\, da - \frac{c}{4\pi}\int_S \phi\, da.$$

Since the last integral is constant,

$$\frac{1}{4\pi}\int_S H(Q, P)\frac{\partial \phi}{\partial n}\, da$$

has at points of S the same normal derivative as ϕ. If there is a function harmonic within S and having a normal derivative equal to $f(Q)$ at points of S, the function

$$\frac{1}{4\pi}\int_S H(Q, P) f(Q)\, da \tag{328}$$

then has the same property. The quantity $H(Q, P)$ is called the **Green's function of the second kind.**

For our purposes it will be sufficient to consider the case of the sphere. In handling this the algebraic work can be greatly reduced by making use of Euler's theorem* on homogeneous functions. This theorem states that if $f(y_1, y_2, \ldots, y_p)$ is a homogeneous function of the nth degree in the variables $y_1, y_2, \ldots, y_p$,

$$y_1 \frac{\partial f}{\partial y_1} + y_2 \frac{\partial f}{\partial y_2} + \cdots + y_p \frac{\partial f}{\partial y_p} = nf.$$

If we substitute

$$x_1 = y_1^{n_1}, \qquad x_2 = y_2^{n_2}, \ldots, \qquad x_p = y_p^{n_p},$$

in this equation we get

$$n_1 x_1 \frac{\partial f}{\partial x_1} + n_2 x_2 \frac{\partial f}{\partial x_2} + \cdots + n_p x_p \frac{\partial f}{\partial x_p} = nf. \tag{329}$$

If $f(x_1, x_2, \ldots, x_p)$ becomes homogeneous of the nth degree when the dimensions $n_1, n_2, \ldots, n_p$ respectively are assigned to the variables, the function then satisfies (329).

Using the notation of Fig. 60 we shall show that

$$H(Q, P) = \frac{1}{r} + \frac{b}{pr'} + \frac{2}{b}\int_b^\infty \left(\frac{b}{pr'} - \frac{1}{b}\right) db \tag{330}$$

is a Green's function of the second kind for the sphere. That this expression is harmonic in the coördinates of Q and that

$$H(Q, P) = H(P, Q)$$

is evident from its form. To show that this function satisfies (326) we consider r and r' as the functions of p, q, b, θ determined by the equations

$$r^2 = p^2 + q^2 - 2pq\cos\theta,$$

$$r'^2 = \frac{b^4}{p^2} + q^2 - 2\frac{b^2}{p} q\cos\theta.$$

Considering b and θ of dimension zero, p and q of dimension 1, the quantity $\frac{1}{r}$ is homogeneous of degree -1. Hence

$$p\frac{\partial}{\partial p}\left(\frac{1}{r}\right) + q\frac{\partial}{\partial q}\left(\frac{1}{r}\right) + \frac{1}{r} = 0.$$

* cf. F. S. Woods, "Advanced Calculus," page 73.

Considering b and θ of dimension zero, q of dimension 1, and p of dimension -1, r' is homogeneous of degree 1. Hence pr', and so also $\frac{b}{pr'}$, is of degree zero, and

$$-p\frac{\partial}{\partial p}\left(\frac{b}{pr'}\right)+q\frac{\partial}{\partial q}\left(\frac{b}{pr'}\right)=0.$$

Considering p and θ of dimension zero, b of dimension $\frac{1}{2}$, and q of dimension 1, the integral

$$2\int_b^\infty\left(\frac{b}{pr'}-\frac{1}{b}\right)db$$

is of degree zero. Hence

$$2q\frac{\partial}{\partial q}\int_b^\infty\left(\frac{b}{pr'}-\frac{1}{b}\right)db-b\left(\frac{b}{pr'}-\frac{1}{b}\right)=0.$$

Dividing the last equation by b and adding to the sum of the two preceding we obtain

$$p\frac{\partial G(Q,P)}{\partial p}+q\frac{\partial H(Q,P)}{\partial q}+G(Q,P)+\frac{1}{b}=0, \qquad (331)$$

where

$$G(Q,P)=\frac{1}{r}-\frac{b}{pr'}$$

is the Green's function for the Dirichlet problem.

Equation (331) is valid for all positions of P and Q (P not coinciding with Q) in the sphere. If Q is on the sphere, $q=b$,

$$G(Q,P)=\frac{\partial}{\partial p}G(Q,P)=0,$$

and so

$$\frac{\partial H(Q,P)}{\partial q}=\frac{\partial H(Q,P)}{\partial n}=-\frac{1}{b^2},$$

which shows that $H(Q,P)$ satisfies equation (326).

To show that (328) is really a solution of the Neumann problem we note first that because of the symmetry of the functions $G(Q,P)$, $H(Q,P)$ in P and Q, the variables p, q may be interchanged in (331) giving

$$q\frac{\partial G(Q,P)}{\partial q}+p\frac{\partial H(Q,P)}{\partial p}+G(Q,P)+\frac{1}{b}=0. \qquad (332)$$

If Q is on the sphere we then have

$$\frac{\partial H(Q, P)}{\partial p} = -\frac{b}{p}\frac{\partial G(Q, P)}{\partial n} - \frac{1}{bp}.$$

If then

$$\phi = \frac{1}{4\pi}\int_S H(Q, P) f(Q)\, da$$

we get

$$\frac{\partial \phi}{\partial p} = -\frac{b}{4\pi p}\int_S f(Q)\frac{\partial G(Q, P)}{\partial n}\, da - \frac{1}{4\pi bp}\int_S f(Q)\, da. \quad (333)$$

Because of the condition (324) the last integral is zero. When P approaches a point Q_0 on the surface, $\frac{\partial \phi}{\partial p}$ approaches the normal derivative of ϕ at that point, p approaches b, and the first integral approaches $f(Q_0)$ as in the Dirichlet problem.

In case of the region outside the sphere, if $H(Q, P)$ is replaced by

$$H(Q, P) - \frac{1}{p}$$

we get

$$\frac{\partial \phi}{\partial p} = -\frac{b}{4\pi p}\int_S f(Q)\frac{\partial G(Q, P)}{\partial n}\, da + \frac{1}{4\pi}\left(\frac{1}{p^2} - \frac{1}{bp}\right)\int_S f(Q)\, da$$

instead of (333). When P approaches the surface, the last term approaches zero and

$$\frac{1}{4\pi}\int_S \left[H(Q, P) - \frac{1}{p}\right] f(Q)\, da$$

is a solution of the Neumann problem whether (324) is satisfied or not.

78. Converse of the Average Value Theorem. In §70 we proved that a harmonic function has at the center of a sphere a value equal to the average of its values on the surface. We shall now prove the converse, namely:

If $\phi(x, y, z)$ is continuous in a closed region V and at the center of each sphere S belonging to V has a value equal to the average of its values on S, then ϕ is harmonic in V.

This shows that the average value property completely char-

acterizes the harmonic functions. To prove the theorem we note first that if two functions ϕ and θ both have this property in a given region, the same is true for

$$\psi = \phi - \theta.$$

Also the argument of §72 shows that unless it is constant a function cannot attain its greatest or least values inside a region where it has this average value property.

Suppose then ϕ has this property in a region V. If P is any interior point we can find a sphere S of center P in V. Let θ be the function harmonic in S and equal to ϕ on S. Our solution of the Dirichlet problem shows there is such a function. Then

$$\psi = \phi - \theta$$

is zero on S. If it is not zero at every point within S, it must attain a greatest or least value. Since ϕ and θ both have the average value property this is impossible. Hence

$$\psi = \phi - \theta = 0$$

at all points inside S. In the neighborhood of each point P within V the function ϕ is therefore identical with some harmonic function. It is therefore harmonic in V.

In this discussion we have assumed that ϕ has at the center of each sphere a value equal to the average of its values on the surface. If the region is one for which there is a solution of the Dirichlet problem the following less restricted theorem is valid:

If $\phi(x, y, z)$ is continuous in a closed region V and at each point P has a value equal to the average of its values on *some* sphere of center P belonging to V, then ϕ is harmonic in V.

For, the Dirichlet problem having a solution in the region, there is a function θ harmonic in V and having upon its boundary the same values as ϕ. The function

$$\psi = \phi - \theta$$

is then zero on the boundary. If it is not zero at all interior points there must be one or more points at which it attains a greatest or least value m. Let P be one of these as near the boundary as possible. There is a sphere S of center P on which the average value of ϕ is $\phi(P)$. On this sphere the average value of ψ is m. Since m is a greatest or least value, ψ must have this

same value at all points of S. But this is impossible since there is no point nearer the boundary than P at which $\psi = m$. Consequently

$$\psi = \phi - \theta = 0$$

at all points of V.

In a region for which the Dirichlet problem has a solution, if a function ϕ is not harmonic there must then be points P such that $\phi(P)$ is not equal to the average value of ϕ on *any* sphere with P as center.

79. Representation of Harmonic Functions as Potentials. The equation

$$\phi(P) = \frac{1}{4\pi}\int_S \left[\frac{1}{r}\frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] da \tag{334}$$

shows that a function ϕ harmonic in a region of boundary S may be regarded as the potential due to a surface charge of density

$$\sigma = \frac{1}{4\pi}\frac{\partial \phi}{\partial n}$$

and a double layer of moment

$$\tau = -\frac{1}{4\pi}\phi$$

spread over S. Assuming the solutions of the Dirichlet and Neumann problems in the region outside S, we can show that ϕ is expressible as the potential of a surface charge or of a double layer alone.

To show this let ϕ', ϕ'' be functions harmonic outside S, the first having upon S the value ϕ, the second having upon S a normal derivative equal to $\frac{\partial \phi}{\partial n}$. If r is drawn from a point P inside S these functions satisfy the equations

$$0 = \frac{1}{4\pi}\int_S \left[\frac{1}{r}\frac{\partial \phi'}{\partial n} - \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] da,$$

$$0 = \frac{1}{4\pi}\int_S \left[\frac{1}{r}\frac{\partial \phi}{\partial n} - \phi'' \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] da.$$

By subtraction from (334) we thus obtain

$$\phi(P) = \frac{1}{4\pi}\int_S \frac{1}{r}\frac{\partial}{\partial n}(\phi - \phi')\,da, \tag{335}$$

$$\phi(P) = \frac{1}{4\pi}\int_S (\phi'' - \phi)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)da. \tag{336}$$

These show that for points inside S, ϕ may be regarded either as the potential due to a surface charge of density

$$\sigma = \frac{1}{4\pi}\frac{\partial}{\partial n}(\phi - \phi'),$$

or as the potential due to a double layer of moment

$$\tau = \frac{1}{4\pi}(\phi'' - \phi),$$

spread over S.

If we let

$$\mathbf{A} = \frac{1}{4\pi}\int_S (\phi'' - \phi)\mathbf{n} \times \nabla\frac{1}{r}da, \tag{337}$$

as in §64 we show that

$$-\nabla\phi = \nabla \times \mathbf{A}. \tag{338}$$

Also by direct operation we get

$$\nabla \cdot \mathbf{A} = 0. \tag{339}$$

Thus the gradient of a harmonic function can be expressed as the curl of a solenoidal vector potential.

80. Characteristics which Determine a Vector. In §66 it was shown that a vector which is irrotational, solenoidal, and continuously differentiable, is the negative gradient of a harmonic function. We can now show that if the vector is merely piecewise continuous the existence and continuity of the derivatives is a consequence of the other conditions. That is,

In a closed simply connected region V a vector $\mathbf{F}(x, y, z)$ which is irrotational, solenoidal, and piecewise continuous, is the negative gradient of a function $\phi(x, y, z)$ which is harmonic.

To prove the theorem we note first that since $\mathbf{F}$ is irrotational, solenoidal, and piecewise continuous, it is continuous in V (§50).

Since the region is simply connected there is then a uniform potential

$$\phi(x, y, z) = \int_P^{P_0} \mathbf{F} \cdot d\mathbf{R}$$

such that

$$\mathbf{F} = -\nabla\phi,$$

and the continuity of ϕ follows from that of $\mathbf{F}$. It remains to show that ϕ is harmonic. To do this let P be any interior point and S a sphere of center P and radius r within the region V. Let

$$f(r) = \int_S \phi \, d\omega,$$

where $d\omega$ is the elementary solid angle subtended at P by the element of surface. Since ϕ has continuous first derivatives

$$\frac{df(r)}{dr} = \int_S \frac{\partial\phi}{\partial r} d\omega = \int_S \frac{\partial\phi}{\partial n} \frac{da}{r^2} = -\int_S \frac{\mathbf{n} \cdot \mathbf{F}\, da}{r^2}.$$

Since r is constant and $\mathbf{F}$ is solenoidal the last integral is zero. Therefore

$$\frac{df(r)}{dr} = 0$$

and so $f(r)$ is constant. By equating the values $f(0)$ and $f(r)$ we obtain

$$4\pi\phi(P) = \int_S \phi \, d\omega.$$

At the center of each sphere the function ϕ thus has a value equal to the average of its values on the surface. Hence ϕ is harmonic in V.

From the preceding theorem and the results of §50 we conclude that in a region which consists of two or more parts a vector is the gradient of a harmonic function if it is continuous in the region and irrotational and solenoidal in each of the parts.

A vector which is irrotational, solenoidal, and piecewise continuous, in the whole of space, is constant.

To show this let S be a sphere of radius r and center at the origin. According to the above theorem there is a function

$\phi(x, y, z)$ harmonic in this sphere such that the vector is $-\nabla\phi$. The components of the vector are thus harmonic in the sphere. Since a piecewise continuous vector is bounded and the sphere can be as large as we please, the components of the vector are bounded and harmonic everywhere. They are therefore constant and so the vector itself is constant.

We say that two vectors have the same discontinuities if their difference is continuous and bounded. The vectors of physics are usually continuous and differentiable except for certain points, curves, and surfaces. *Such a vector is determined in a finite region by its divergence, curl, discontinuities, and normal component on the boundary.*

By this we mean that two vectors $\mathbf{F}_1$, $\mathbf{F}_2$ are equal if they agree in these respects. This follows because the points at which the vectors or their first derivatives are discontinuous divide the region into a finite number of parts in each of which $\mathbf{F}_2 - \mathbf{F}_1$ is irrotational and solenoidal. Since it is also continuous,

$$\mathbf{F}_2 - \mathbf{F}_1 = \nabla\phi,$$

where ϕ is harmonic in the region. Since the normal component of $\nabla\phi$ is zero on the boundary, ϕ is constant, and so

$$\mathbf{F}_2 - \mathbf{F}_1 = 0.$$

If the vector is defined at all points of space it is determined, except for an added constant, by its divergence, curl, and discontinuities.

PROBLEMS

1. Show that

$$\frac{x}{r^3}$$

is a solution of Laplace's equation.

2. If each component of $\mathbf{F}$ is a harmonic function, show that

$$\nabla^2\mathbf{F} = 0.$$

3. If ϕ and $f(\phi)$ are both harmonic in a given region and ϕ is not constant, show that $f(\phi)$ is a linear function of ϕ.

4. If $f(x)$ is an analytic function of x and

$$f(x + y\sqrt{-1}) = u + v\sqrt{-1},$$

prove that u and v are harmonic functions.

5. In spherical coördinates show that Laplace's equation is

$$\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial \phi}\right) = 0.$$

Find solutions of the forms $f(r)$, $f(\theta)$, and $f(\phi)$.

6. Let r_1, r_2 be the distances between the points P_1, P_2 and the element of area da on a closed surface S. Find the value of

$$\int_S \left[\frac{1}{r_1}\frac{\partial}{\partial n}\left(\frac{1}{r_2}\right) - \frac{1}{r_2}\frac{\partial}{\partial n}\left(\frac{1}{r_1}\right)\right] da,$$

(1) when P_1 and P_2 are both inside S,
(2) when P_1 is inside and P_2 outside,
(3) when P_1 and P_2 are both outside.

7. If $r = PQ$, show that the average value of

$$\frac{1}{r}$$

for all positions of Q on a surface S is the electrostatic potential at P due to a charge of constant density σ on the surface. In this way, or otherwise, show that if S is a sphere of radius b and c is the distance between P and its center, the average value is

$$\frac{1}{c}$$

if P lies outside the sphere and

$$\frac{1}{b}$$

if inside.

8. If ϕ_1 and ϕ_2 are harmonic within a closed surface S and ϕ_1 is constant on S, show that

$$\int_S \phi_2 \frac{\partial \phi_1}{\partial n} da = 0.$$

9. If a closed surface S is an equipotential for a system of charges which it encloses, show that the potential ϕ at points outside S is the same as that due to a surface charge on S of density

$$\sigma = -\frac{1}{4\pi}\frac{\partial \phi}{\partial n}.$$

10. If a harmonic function is positive at all points on the boundary of a region, show that it is positive at all points inside.

11. The potential

$$\phi = \int_S \frac{\sigma\, da}{r}$$

due to a charge on a closed surface S is zero at all points inside S. Show that σ is zero.

12. The potential

$$\phi = \int_V \frac{\rho\, dv}{r} + \int_S \frac{\sigma\, da}{r}$$

due to a space charge in a region V and a surface charge on its boundary S is zero at all points of the region. Show that ρ and σ are zero.

13. If ϕ is harmonic in a region, c is constant, and

$$(\phi - c)\frac{\partial \phi}{\partial n} \leqq 0$$

at all points of the boundary, show that ϕ is constant in the region.

14. Prove that if ϕ is harmonic in a region, constant on part of the boundary, and its normal derivative is zero on the remainder of the boundary, it is constant in the region.

15. The components of a vector $\mathbf{F}$ are harmonic functions. If $\nabla \times \mathbf{F}$ is zero at all points of a closed surface, show that it is zero at all points inside.

16. Show that

$$\frac{1}{r}$$

is the Green's function for the region consisting of all space.

17. Let P, P' be points symmetrical with respect to a plane and r, r' their distances from Q. Show that

$$G(Q, P) = \frac{1}{r} - \frac{1}{r'}$$

is the Green's function for the region on one side of the plane. Verify equations (310) and (312) for this case.

18. Let $f(x, y, z)$ be continuous at points of a plane S and let r be the distance from $P(x, y, z)$ to the element of area da. Show that

$$\phi(x, y, z) = -\frac{1}{2\pi}\int_S f \frac{\partial}{\partial n}\left(\frac{1}{r}\right) da$$

is the solution of the Dirichlet problem for the region on one side of S.

19. Let $P(x, y, z)$, $P'(x', y', z')$ be inverse with respect to a sphere of radius b and r, r' their distances from its center. To points P of a region V correspond points P' of a region V'. If $\phi(x, y, z)$ is harmonic in the region V, show that

$$\phi'(x', y', z') = \frac{b}{r'}\phi,$$

considered as a function of x', y', z', is harmonic in the region V'.

20. Find the function ϕ' in the preceding problem if $\phi = x$ and the center of the sphere is the origin.

21. Show that

$$H(Q, P) = \frac{1}{r} + \frac{1}{r'}$$

is the Green's function of the second kind and

$$\phi = \frac{1}{2\pi}\int_S \frac{f(Q)\, da}{r}$$

is the solution of the Neumann problem for the region of problem 17.

22. In the region outside a closed surface a harmonic function is bounded and at all finite points of this region it is harmonic. Prove that the function has a limit at infinity.

CHAPTER VIII

SCALAR AND VECTOR POTENTIALS

81. Introduction. In §§45 and 48 we found that an irrotational vector is the negative gradient of a scalar potential and a solenoidal vector is the curl of a vector potential. In §83 we shall show that any vector with sufficient continuity is divisible into irrotational and solenoidal parts, and so is expressible in terms of a scalar and a vector potential.

The fundamental problem in the analysis of a vector field is the determination of these potentials and their expression in terms of the essential characteristics of the vector, namely, divergence, curl, discontinuities, and boundary values. For when the potentials are known the vector itself can be determined by differentiation.

82. Expression of a Scalar Function as Potential. First we shall show that in a region V where its second derivatives are piecewise continuous a given scalar function $\phi(x, y, z)$ can be expressed as a scalar potential. In such a region ϕ is characterized by boundary values and the quantities ρ, σ, τ defined as follows:

(1) The region can be divided into sub-regions in each of which $\phi(x, y, z)$ has continuous second derivatives. In each of these sub-regions let $\rho(x, y, z)$ be the function defined by the equation

$$\nabla^2\phi = -4\pi\rho.$$

(2) In the region there may be certain surfaces S_1 on opposite sides of which the normal derivative $\frac{\partial\phi}{\partial n}$ has different values. At a point of such a surface let $\frac{\partial\phi}{\partial n_+}$, $\frac{\partial\phi}{\partial n_-}$ be the derivatives taken in the direction of the unit normal n on the positive and on the negative side respectively, and σ the function defined by the equation

$$\frac{\partial\phi}{\partial n_+} - \frac{\partial\phi}{\partial n_-} = -4\pi\sigma.$$

(3) In the region there may be surfaces S_2 on which ϕ is discontinuous. At a point of such a surface let ϕ_+, ϕ_- be the values on the positive and on the negative side respectively, and τ the function defined by the equation

$$\phi_+ - \phi_- = 4\pi\tau.$$

If S is the boundary of V and r is the distance between $P(x, y, z)$ and the element of integration, the function

$$f(x, y, z) = \int_V \frac{\rho\, dv}{r} + \int_{S_1} \frac{\sigma\, da}{r} + \int_{S_2} \tau \frac{\partial}{\partial n}\left(\frac{1}{r}\right) da + \frac{1}{4\pi}\int_S \left[\frac{1}{r}\frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] da \qquad (340)$$

has the value $\phi(x, y, z)$ at interior points where ϕ is continuous and the value zero at points outside the region V.

The right side of (340) is the electric potential due to a space charge of density ρ extending through the region, a surface charge of density σ on S_1, a double layer of moment τ on S_2, and a surface charge of density

$$\frac{1}{4\pi}\frac{\partial \phi}{\partial n}$$

and double layer of moment

$$-\frac{\phi}{4\pi}$$

on the boundary. The above theorem states that this distribution has a potential ϕ at points inside V and zero at points outside.

To prove the theorem let P be an interior point not on S_1 or S_2, and S_0 a sphere of center P so small that it contains no point of S_1, S_2, or S. In the region V' between S_0 and S the vector

$$\mathbf{U} = \frac{1}{r}\nabla\phi - \phi\nabla\frac{1}{r} \qquad (341)$$

is piecewise differentiable and

$$\nabla \cdot \mathbf{U} = \frac{1}{r}\nabla^2\phi = -\frac{4\pi\rho}{r}$$

in each sub-region where ϕ has continuous second derivatives. If we consider both sides of S_1 and S_2 as belonging to the bound-

ary, $\mathbf{U}$ will also be continuous in V'. By the divergence theorem we then have

$$\int_{S'} \mathbf{U} \cdot \mathbf{n}\, da = -4\pi \int_{V'} \frac{\rho\, dv}{r}, \tag{342}$$

the surface integral being taken over S_0, S, and both sides of S_1 and S_2 (cf. page 70).

At points of S_0 the normal is directed toward P and r is constant. Hence

$$\int_{S_0} \mathbf{U} \cdot \mathbf{n}\, da = -\int_{S_0} r \frac{\partial \phi}{\partial n} d\omega - \int_{S_0} \phi\, d\omega,$$

where $d\omega$ is the solid angle subtended by da at P. When the radius of S_0 approaches zero the first integral on the right approaches zero and the second approaches $-4\pi\phi(P)$. Consequently (342) gives as limit

$$\int \mathbf{U} \cdot \mathbf{n}\, da - 4\pi\phi(P) = -4\pi \int_V \frac{\rho\, dv}{r}, \tag{343}$$

the surface integral on the left being taken over S and both sides of S_1 and S_2.

On the surface S_1 let $\mathbf{n}$ be the unit normal, ϕ'', $\nabla\phi''$ limiting values on the side toward which $\mathbf{n}$ extends, and ϕ', $\nabla\phi'$ limiting values on the other side. In integrating over either side of S_1 the normal is directed outward from V', that is, toward the surface. The result of integrating over both sides is

$$\int_{S_1} \mathbf{U} \cdot \mathbf{n}\, da = \int_{S_1} \mathbf{n} \cdot \left(\frac{1}{r}\nabla\phi' - \phi'\nabla\frac{1}{r}\right) da - \int_{S_1} \mathbf{n} \cdot \left(\frac{1}{r}\nabla\phi'' - \phi''\nabla\frac{1}{r}\right) da.$$

Since ϕ is continuous on S_1 and

$$\mathbf{n} \cdot (\nabla\phi'' - \nabla\phi') = \frac{\partial\phi}{\partial n_+} - \frac{\partial\phi}{\partial n_-} = -4\pi\sigma,$$

this is equal to

$$4\pi \int_{S_1} \frac{\sigma\, da}{r}.$$

In a similar way, by integrating over S_2 and using the conditions

$$\mathbf{n} \cdot (\nabla\phi'' - \nabla\phi') = 0, \qquad \phi'' - \phi' = 4\pi\tau,$$

we obtain

$$4\pi\int_{S_2}\tau\frac{\partial}{\partial n}\left(\frac{1}{r}\right)da.$$

Introducing these values for the integrals over S_1 and S_2 and solving for $\phi(P)$ equation (343) gives

$$\phi(P) = \int_V \frac{\rho\,dv}{r} + \int_{S_1}\frac{\sigma\,da}{r} + \int_{S_2}\tau\frac{\partial}{\partial n}\left(\frac{1}{r}\right)da + \frac{1}{4\pi}\int_S\left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]da. \tag{344}$$

In proving this we have assumed that P was not a point of S_1 or S_2. Allowing P to approach one of these surfaces it is clear from the nature of the integrals that the formula is also valid in the limit for a point on S_1, but not for a point where τ is different from zero on S_2.

If P lies outside the region V, the sphere S_0 is not needed and the above argument gives

$$0 = \int_V \frac{\rho\,dv}{r} + \int_{S_1}\frac{\sigma\,da}{r} + \int_{S_2}\tau\frac{\partial}{\partial n}\left(\frac{1}{r}\right)da + \frac{1}{4\pi}\int_S\left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]da. \tag{345}$$

The boundary integral in (344) represents a function harmonic within the region V. Therefore

$$\phi = \int_V \frac{\rho\,dv}{r} + \int_{S_1}\frac{\sigma\,da}{r} + \int_{S_2}\tau\frac{\partial}{\partial n}\left(\frac{1}{r}\right)da + \phi_1, \tag{346}$$

where ϕ_1 is harmonic in this region. If ϕ is regular at infinity, by taking for S a sphere of center P it is easy to show that ϕ_1 approaches zero when the radius of this sphere increases indefinitely. We then have

$$\phi = \int_V \frac{\rho\,dv}{r} + \int_{S_1}\frac{\sigma\,da}{r} + \int_{S_2}\tau\frac{\partial}{\partial n}\left(\frac{1}{r}\right)da, \tag{347}$$

the integrals being taken over all values of ρ, σ, τ. Equations (344) and (345) are merely this equation for the function which is equal to ϕ at points within the region V and is zero at points outside.

83. Expression of a Vector as the Sum of an Irrotational and a Solenoidal Vector. In a finite region V let

$$\mathbf{F} = \mathbf{i}f_1 + \mathbf{j}f_2 + \mathbf{k}f_3$$

be piecewise differentiable. At points where $\mathbf{F}$ is continuous its components then satisfy Hölder conditions (§59).

With each point $P(x, y, z)$ of space associate the vector

$$\mathbf{W}(P) = \int_V \frac{\mathbf{F}\, dv}{r}, \tag{348}$$

where r is the distance between P and the element of integration. The x-component of $\mathbf{W}$, namely

$$w_1 = \int_V \frac{f_1\, dv}{r},$$

is equal to the electrostatic potential due to a charge of density f_1 in V. It therefore satisfies Poisson's equation

$$\nabla^2 w_1 = -4\pi f_1$$

at interior points where f_1 is continuous, and Laplace's equation

$$\nabla^2 w_1 = 0$$

at exterior points. Similar equations are satisfied by the other components. Multiplying these equations by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively and adding we have

$$\nabla^2 \mathbf{W} = -4\pi \mathbf{F} \tag{349}$$

at interior points where $\mathbf{F}$ is continuous, and

$$\nabla^2 \mathbf{W} = 0 \tag{350}$$

at points outside the region V.

Let $\phi(x, y, z)$ be the scalar and $\mathbf{A}(x, y, z)$ the vector defined by the equations

$$4\pi\phi = \nabla \cdot \mathbf{W} = -\int_V \mathbf{F} \cdot \nabla \frac{1}{r}\, dv, \tag{351}$$

$$4\pi\mathbf{A} = \nabla \times \mathbf{W} = \int_V \mathbf{F} \times \nabla \frac{1}{r}\, dv. \tag{352}$$

In the integrals there is a change of sign due to the usual shift from differentiation with respect to the coördinates of P to

differentiation with respect to those of Q. By discussions similar to those of §§56 to 59 it can be shown that these functions are everywhere definite and continuous and are differentiable at interior points where $\mathbf{F}$ is continuous. Equation (67) gives

$$\nabla \times (\nabla \times \mathbf{W}) = \nabla(\nabla \cdot \mathbf{W}) - \nabla^2\mathbf{W}.$$

Substituting the above values for $\nabla^2\mathbf{W}$, $\nabla \cdot \mathbf{W}$, $\nabla \times \mathbf{W}$ in this equation and solving for $\mathbf{F}$, we have

$$\mathbf{F} = -\nabla\phi + \nabla \times \mathbf{A} \tag{353}$$

at interior points where $\mathbf{F}$ is continuous, and

$$0 = -\nabla\phi + \nabla \times \mathbf{A} \tag{354}$$

at points outside the region V.

Since $\nabla\phi$ is irrotational and $\nabla \times \mathbf{A}$ is solenoidal, equation (353) expresses that $\mathbf{F}$ *is the sum of two vectors, one of which is irrotational, the other solenoidal.* That such a resolution is possible is sometimes called **Helmholtz's theorem.**

84. Infinite Regions. In the preceding discussion we have considered values of $\mathbf{F}$ only in a finite region. To obtain similar results for an infinite region we must place some restriction on values of $\mathbf{F}$ at a great distance from the origin.

The vector $\mathbf{F}$ is called **regular at infinity** if two positive numbers m, c exist such that

$$r_0^2|\mathbf{F}(\xi, \eta, \zeta)| < m \tag{355}$$

whenever

$$r_0 = \sqrt{\xi^2 + \eta^2 + \zeta^2} \tag{356}$$

is greater than c.

If $\mathbf{F}$ *is piecewise differentiable everywhere and regular at infinity, the integrals*

$$\phi(P) = -\frac{1}{4\pi}\int \mathbf{F} \cdot \nabla\frac{1}{r}\,dv, \tag{357}$$

$$\mathbf{A}(P) = \frac{1}{4\pi}\int \mathbf{F} \times \nabla\frac{1}{r}\,dv, \tag{358}$$

taken over all space, converge and satisfy the equation

$$\mathbf{F} = -\nabla\phi + \nabla \times \mathbf{A} \tag{359}$$

at points where $\mathbf{F}$ *is continuous.*

To show this let $P(x, y, z)$ be any point and S a sphere with center at the origin and radius greater than c and also greater than twice the distance between P and the origin. The distance from P to a point $Q(\xi, \eta, \zeta)$ outside this sphere is then greater than $\frac{1}{2}r_0$. For such a pair of points P, Q

$$\left|\mathbf{F}\cdot\nabla\frac{1}{r}\right| \leqq |\mathbf{F}|\frac{1}{r^2} < \frac{4m}{r_0^4}.$$

The part of ϕ outside S therefore has a numerical value less than

$$\frac{m}{\pi}\int\frac{dv}{r_0^4} < 4m\int_c^\infty \frac{dr_0}{r_0^2} = \frac{4m}{c},$$

and this approaches zero when c tends to infinity. Hence ϕ converges. In a similar way we show that $\mathbf{A}$ converges. By a discussion analogous to that for the finite region we can show that ϕ and $\mathbf{A}$ are everywhere continuous and, at points where $\mathbf{F}$ is continuous, are differentiable.

To show that they satisfy (359) let ϕ', $\mathbf{A}'$ be the parts of ϕ, $\mathbf{A}$ in a sphere S with center at the origin. These are the potentials for the finite region within S. From (353) we therefore have

$$\mathbf{F} = -\nabla\phi' + \nabla\times\mathbf{A}'$$

at points where $\mathbf{F}$ is continuous. The differences of ϕ, $\mathbf{A}$ and ϕ', $\mathbf{A}'$ are integrals over the region outside S. Since these integrals and their derivatives converge, the derivatives of ϕ' and $\mathbf{A}'$ approach those of ϕ and $\mathbf{A}$ as limits when the radius of S increases indefinitely. Hence this equation gives

$$\mathbf{F} = -\nabla\phi + \nabla\times\mathbf{A}$$

as limit.

85. Potentials of a Vector. The functions ϕ, $\mathbf{A}$ determined by equations (351), (352) or (357), (358) may be considered as a scalar and a vector potential associated with a vector $\mathbf{F}$ and a region. For a piecewise differentiable vector these potentials have the following properties:

(1) They are continuous and piecewise differentiable everywhere.

(2) They have the limit zero at infinity.

(3) The vector $\mathbf{A}$ is solenoidal.

If $\mathbf{F}$ is piecewise continuous everywhere and (359) is satisfied at all points where $\mathbf{F}$ is continuous, these conditions uniquely define the potentials.

To show this suppose ϕ, $\mathbf{A}$ and ϕ', $\mathbf{A}'$ satisfy the above conditions and

$$\mathbf{F} = -\nabla\phi + \nabla \times \mathbf{A} = -\nabla\phi' + \nabla \times \mathbf{A}'$$

at all points where $\mathbf{F}$ is continuous. The vector

$$\nabla(\phi' - \phi) = \nabla \times (\mathbf{A}' - \mathbf{A})$$

is then irrotational, solenoidal, and piecewise continuous everywhere. Hence

$$\phi' - \phi = 0,$$

since it is harmonic everywhere and zero at infinity (§71). It follows that

$$\nabla \times (\mathbf{A}' - \mathbf{A}) = 0$$

and so

$$\mathbf{A}' - \mathbf{A} = 0,$$

since it is irrotational, solenoidal, continuous, and zero at infinity (§80).

In a finite region the potentials are however not uniquely determined. For a vector $\mathbf{F}'$ can be defined which has the same values as $\mathbf{F}$ in a region V and different values outside. Its potentials satisfy equation (353) in the region V but not outside. Equations (353), (354) show that the potentials determined by (351), (352) are those of the vector which is equal to $\mathbf{F}$ at points within the region V and zero at points outside.

If there are points where $\mathbf{F}$ or its first derivatives are not bounded the potentials are not unique even when (359) is satisfied everywhere. For if $\nabla\phi$ becomes infinite at a particular point Q we can add to ϕ the potential ϕ' of a doublet at Q and to $\mathbf{A}$ the vector potential $\mathbf{A}'$ of the same doublet (cf. §64). The quantities ϕ', $\mathbf{A}'$, are everywhere continuous and differentiable except at Q. Since

$$\nabla\phi' = \nabla \times \mathbf{A}'$$

the equation

$$-\nabla\phi + \nabla \times \mathbf{A} = -\nabla(\phi + \phi') + \nabla \times (\mathbf{A} + \mathbf{A}')$$

is therefore satisfied at all points where $\nabla\phi$ and $\nabla \times \mathbf{A}$ are continuous.

86. Expression of the Potentials in Terms of Characteristics of the Vector. By means of equation (353) a vector which is piecewise differentiable in a region V can be expressed in terms of a scalar potential

$$\phi(P) = \frac{1}{4\pi}\nabla \cdot \mathbf{W} = -\frac{1}{4\pi}\int_V \mathbf{F} \cdot \nabla \frac{1}{r}\, dv \tag{360}$$

and a vector potential

$$\mathbf{A}(P) = \frac{1}{4\pi}\nabla \times \mathbf{W} = \frac{1}{4\pi}\int_V \mathbf{F} \times \nabla \frac{1}{r}\, dv. \tag{361}$$

In the given region the vector $\mathbf{F}$ is continuous except that on certain surfaces S_1 there is a finite increment $\Delta\mathbf{F}$ in $\mathbf{F}(x, y, z)$ when the point (x, y, z) crosses the surface in the positive direction of the normal. Let ρ, $\mathbf{I}$, σ, $\mathbf{J}$ be quantities defined by the equations

$$\nabla \cdot \mathbf{F} = 4\pi\rho, \tag{362}$$

$$\nabla \times \mathbf{F} = 4\pi\mathbf{I}, \tag{363}$$

$$\mathbf{n} \cdot \Delta\mathbf{F} = 4\pi\sigma, \tag{364}$$

$$\mathbf{n} \times \Delta\mathbf{F} = 4\pi\mathbf{J}. \tag{365}$$

For such a vector the potentials have the values

$$\phi(P) = \int_V \frac{\rho\, dv}{r} + \int_{S_1} \frac{\sigma\, da}{r} - \frac{1}{4\pi}\int_S \frac{\mathbf{n} \cdot \mathbf{F}\, da}{r}, \tag{366}$$

$$\mathbf{A}(P) = \int_V \frac{\mathbf{I}\, dv}{r} + \int_{S_1} \frac{\mathbf{J}\, da}{r} - \frac{1}{4\pi}\int_S \frac{\mathbf{n} \times \mathbf{F}\, da}{r}, \tag{367}$$

where r is the distance between P and the element of integration, and S is the surface forming the complete boundary of V.

To prove (366) we substitute

$$\mathbf{F} \cdot \nabla \frac{1}{r} = \nabla \cdot \left(\frac{\mathbf{F}}{r}\right) - \frac{1}{r}\nabla \cdot \mathbf{F}$$

in (360) and replace $\nabla \cdot \mathbf{F}$ by $4\pi\rho$, thus obtaining

$$\phi(P) = \int_V \frac{\rho\, dv}{r} - \frac{1}{4\pi}\int_V \nabla \cdot \left(\frac{\mathbf{F}}{r}\right) dv. \tag{368}$$

Let S_0 be a small sphere of center P and radius r_0. In the region V' having for boundary S_0, S, and both sides of S_1, the function $\frac{\mathbf{F}}{r}$ has continuous first derivatives. Consequently

$$\int_{V'} \nabla \cdot \left(\frac{\mathbf{F}}{r}\right) dv = \int_{S'} \frac{\mathbf{n} \cdot \mathbf{F}\, da}{r},$$

where S' is the complete boundary of V'. Since

$$\int_{S_0} \frac{\mathbf{n} \cdot \mathbf{F}\, da}{r}$$

approaches zero with r_0, this has for limit

$$\int_{V} \nabla \cdot \left(\frac{\mathbf{F}}{r}\right) dv = \int \frac{\mathbf{n} \cdot \mathbf{F}\, da}{r},$$

the surface integral being taken over S and both sides of S_1. In integrating over S_1 the normals are directed outward from V', that is, toward the surface. If $\mathbf{n}$ is the normal to the surface, $\mathbf{F}''$ the value of $\mathbf{F}$ on the side toward which it extends, and $\mathbf{F}'$ the value on the other side, the integral over both sides of S_1 is

$$-\int_{S_1} \frac{\mathbf{n} \cdot (\mathbf{F}'' - \mathbf{F}')\, da}{r} = -\int_{S_1} \frac{\mathbf{n} \cdot \Delta\mathbf{F}\, da}{r} = -4\pi \int_{S_1} \frac{\sigma\, da}{r}.$$

Introducing these values equation (368) takes the form (366).

From equation (361) and the identity

$$\mathbf{F} \times \nabla \frac{1}{r} = -\nabla \times \left(\frac{\mathbf{F}}{r}\right) + \frac{1}{r} \nabla \times \mathbf{F},$$

by a similar procedure we prove (367).

If the vector $\mathbf{F}$ is piecewise differentiable everywhere and regular at infinity, when S expands indefinitely the boundary integrals in (366) and (367) approach zero and these equations give as limits

$$\phi(P) = \int \frac{\rho\, dv}{r} + \int \frac{\sigma\, da}{r}, \tag{369}$$

$$\mathbf{A}(P) = \int \frac{\mathbf{I}\, dv}{r} + \int \frac{\mathbf{J}\, da}{r}, \tag{370}$$

the integrals being taken over all values of ρ, σ, $\mathbf{I}$, $\mathbf{J}$. Equations (366) and (367) are merely these equations for the vector which

is equal to $\mathbf{F}$ at points within the region V and zero at points outside.

87. Equation of Continuity. The vectors $\mathbf{I}$ and $\mathbf{J}$ are not arbitrary. They are subject to the equation of continuity

$$\nabla \cdot \mathbf{A} = 0. \tag{371}$$

The effect of this condition is to make $\mathbf{I}$ *and* $\mathbf{J}$ *behave like space and surface currents of something which is nowhere created or destroyed.* In the electromagnetic field they usually represent currents of electricity.

To show the general connection between this conservative character of $\mathbf{I}$ and $\mathbf{J}$ and the solenoidal property of $\mathbf{A}$, we shall calculate the divergence of $\mathbf{A}$. For simplicity we take the vector potential in the form (370). If the vector $\mathbf{F}$ is defined only in a limited region, it can be replaced by another which has the value $\mathbf{F}$ at points within the region and zero at points outside. Differentiation gives

$$\nabla \cdot \mathbf{A} = -\int \mathbf{I} \cdot \nabla \frac{1}{r} dv - \int \mathbf{J} \cdot \nabla \frac{1}{r} da, \tag{372}$$

the negative signs being due to the change from differentiation with respect to the coördinates of P to differentiation with respect to those of Q. The integrations are taken over all values of $\mathbf{I}$ and $\mathbf{J}$ defined by (363) and (365). We assume that $\mathbf{F}$ and $\mathbf{I}$ are piecewise differentiable in space, and that $\mathbf{J}$ is piecewise differentiable on each surface where $\mathbf{F}$ has discontinuities.

If there are surfaces on which $\mathbf{I}$ has discontinuities, let $\Delta\mathbf{I}$ be the increment in $\mathbf{I}(x, y, z)$ when the point (x, y, z) crosses the surface in the positive direction of the normal. By an integration like that used for ϕ in §86 we obtain

$$-\int \mathbf{I} \cdot \nabla \frac{1}{r} dv = \int \frac{\nabla \cdot \mathbf{I}\, dv}{r} + \int \frac{\mathbf{n} \cdot \Delta\mathbf{I}\, da}{r}, \tag{373}$$

the integrals being extended respectively through each region in which $\mathbf{I}$ is defined and over each surface on which $\mathbf{I}$ is discontinuous.

Since $\mathbf{J}$ is perpendicular to $\mathbf{n}$, by equation (185)

$$\nabla \cdot \mathbf{J} = \mathbf{n} \cdot \nabla \times (\mathbf{n} \times \mathbf{J}).$$

The identity

$$-\mathbf{n}\times(\mathbf{J}\times\mathbf{n})\cdot\nabla\frac{1}{r}=-\frac{\mathbf{n}\cdot\nabla\times(\mathbf{J}\times\mathbf{n})}{r}+\mathbf{n}\cdot\nabla\times\left(\frac{\mathbf{J}\times\mathbf{n}}{r}\right)$$

is thus equivalent to

$$-\mathbf{J}\cdot\nabla\frac{1}{r}=\frac{\nabla\cdot\mathbf{J}}{r}+\mathbf{n}\cdot\nabla\times\left(\frac{\mathbf{J}\times\mathbf{n}}{r}\right) \tag{374}$$

If $\mathbf{J}$ is piecewise differentiable on a particular surface, there may be a finite number of curves on that surface where $\mathbf{J}$ is discontinuous. Assign a positive direction along each of those curves and let $\Delta\mathbf{J}$ be the increment in $\mathbf{J}(\xi, \eta, \zeta)$ when the point (ξ, η, ζ) crosses one of the curves from a portion of the surface along the border of which this direction is positive. By a discussion similar to that for space we get

$$-\int\mathbf{J}\cdot\nabla\frac{1}{r}da=\int\frac{\nabla\cdot\mathbf{J}\,da}{r}+\int\frac{\mathbf{n}\times\Delta\mathbf{J}\cdot d\mathbf{R}}{r}, \tag{375}$$

the integrals being taken over all values of $\mathbf{J}$ and $\Delta\mathbf{J}$.

The sum of (373) and (375) is

$$\nabla\cdot\mathbf{A}=\int\frac{\nabla\cdot\mathbf{I}\,dv}{r}+\int\frac{(\mathbf{n}\cdot\Delta\mathbf{I}+\nabla\cdot\mathbf{J})da}{r}+\int\frac{\mathbf{n}\times\Delta\mathbf{J}\cdot d\mathbf{R}}{r}, \tag{376}$$

the integrals being taken over all values of the respective quantities. These integrals represent the electrostatic potentials of a space charge, a surface charge, and a linear charge respectively. The equation

$$\oint\mathbf{E}\cdot\mathbf{n}\,da=4\pi\,(\text{charge enclosed})$$

shows that if the potential is everywhere zero the charges are everywhere zero. If $\mathbf{A}$ is everywhere solenoidal, the three equations

$$\nabla\cdot\mathbf{I}=0, \tag{377}$$

$$\mathbf{n}\cdot\Delta\mathbf{I}+\nabla\cdot\mathbf{J}=0, \tag{378}$$

$$\mathbf{n}\times\Delta\mathbf{J}\cdot d\mathbf{R}=0 \tag{379}$$

must then be everywhere satisfied.

Considering $\mathbf{I}$ and $\mathbf{J}$ as representing currents, equation (377) expresses that the amount which flows out from a small region

is equal to the amount which flows in (cf. §33). Equation (378) expresses that the total flow from a portion of a conducting surface into space and along the surface (cf. equation (186)) is zero. Equation (379) expresses that the flow from one sub-region across a curve on a conducting surface is equal to the flow into the adjacent sub-region. These three equations thus express that **I** and **J**, considered as space and surface currents, represent a flow of something which is conserved. For **I** and **J** to have this property the above discussion shows it is necessary and sufficient that **A** be everywhere solenoidal.

88. Boundary Integrals. We again consider **F** as piecewise differentiable everywhere and regular at infinity, the case of a finite region being covered by using a vector which equals **F** in that region and is zero outside.

Let S be the surface forming the complete boundary of any region, and S_1 the surface or set of surfaces in that region on which **F** has discontinuities. For the region bounded by S and both sides of S_1 (cf. page 70), the divergence theorem gives

$$\int_S \mathbf{F}\cdot\mathbf{n}\,da + \int_{S_1} \mathbf{F}\cdot\mathbf{n}\,da = \int_V \nabla\cdot\mathbf{F}\,dv.$$

From (362) and (364) this is equivalent to

$$\int_S \mathbf{F}\cdot\mathbf{n}\,da = 4\pi\int \rho\,dv + 4\pi\int \sigma\,da, \tag{380}$$

the integrals on the right being taken over all values of ρ and σ in the region of boundary S.

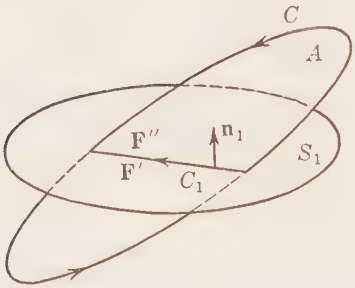


FIG. 61

Let C be the curve forming the complete boundary of an area A, and C_1 the curve or set of curves in which that area intersects the surface S_1 on which **F** is discontinuous. For the region on A bounded by C and both sides of C_1, Stokes's theorem gives

$$\int_C \mathbf{F}\cdot d\mathbf{R} + \int_{C_1} \mathbf{F}\cdot d\mathbf{R} = \int_A \mathbf{n}\cdot\nabla\times\mathbf{F}\,da. \tag{381}$$

Let $\mathbf{n}_1$ be the normal to S_1 and

$$\Delta\mathbf{F} = \mathbf{F}'' - \mathbf{F}'$$

the increment of $\mathbf{F}$ defined as in §86. On C_1 take $d\mathbf{R}$ in the direction which is positive for points of A on the side where $\mathbf{F}'$ is measured. From equation (365), since $d\mathbf{R}$ is perpendicular to $\mathbf{n}_1$, the integral along C_1 has the value

$$-\int \Delta\mathbf{F} \cdot d\mathbf{R} = -4\pi \int \mathbf{J} \times \mathbf{n}_1 \cdot d\mathbf{R}.$$

Consequently (381) is equivalent to

$$\int_C \mathbf{F} \cdot d\mathbf{R} = 4\pi \int \mathbf{I} \cdot \mathbf{n}\, da + 4\pi \int \mathbf{J} \times \mathbf{n}_1 \cdot d\mathbf{R}, \tag{382}$$

the integrals on the right being taken over values of $\mathbf{I}$ and $\mathbf{J}$ in the area of boundary C.

If we interpret ρ, σ and $\mathbf{I}$, $\mathbf{J}$ as charges and currents respectively, equations (380) and (382) can be written

$$\oint \mathbf{F} \cdot \mathbf{n}\, da = 4\pi \text{ (total charge enclosed)}, \tag{383}$$

$$\oint \mathbf{F} \cdot d\mathbf{R} = 4\pi \text{ (total current linked)}. \tag{384}$$

In the last equation the curve around which the integral is taken determines a positive direction along the normal to any surface having that curve as boundary. By the current linked we mean the total current which crosses such a surface, the flow being considered positive when in the positive direction of the normal.

If the vector $\mathbf{F}$ is piecewise differentiable everywhere and regular at infinity, it is uniquely determined by equations (380) and (382). For if two vectors satisfy these conditions their difference, being irrotational, solenoidal, piecewise continuous, and zero at infinity, is zero.

89. Irrotational Vectors. Solenoidal Vectors. The equation

$$\mathbf{F} = -\nabla\phi + \nabla \times \mathbf{A}$$

and the values (366), (367) or (369), (370) for ϕ and $\mathbf{A}$ enable us to express a vector $\mathbf{F}$ in terms of two potentials. If the vector

is irrotational or solenoidal we know that one of these potentials is sufficient. The other should then be zero.

First suppose the vector is irrotational. In this case (§§45, 50)

$$\nabla \times \mathbf{F} = 0, \qquad \mathbf{n} \times \Delta \mathbf{F} = 0.$$

Equations (363) and (365) show that $\mathbf{I}$ and $\mathbf{J}$ are then zero. If these conditions hold everywhere and the vector is regular at infinity, equation (370) shows that $\mathbf{A}$ is zero and so

$$\mathbf{F} = -\nabla\phi.$$

To obtain a similar expression for a vector which is only defined in a limited region, we extend the vector outside that region so that it becomes irrotational everywhere and regular at infinity. This can be done as follows. Since $\mathbf{F}$ is irrotational in the region it is the negative gradient of a function ϕ. At points outside the region let $\mathbf{F}$ have the value $-\nabla\phi'$, where ϕ' is harmonic outside and has upon the boundary the same values as ϕ. The vector thus extended is irrotational everywhere, since it is irrotational inside the region and outside and has a tangential component continuous on the boundary (§50). It is also regular at infinity. For it the vector $\mathbf{A}$ is therefore zero.

If the vector is solenoidal (§§48, 50)

$$\nabla \cdot \mathbf{F} = 0, \qquad \mathbf{n} \cdot \Delta \mathbf{F} = 0.$$

Equations (362) and (364) show that ρ and σ are then zero. If these conditions hold at all points of space equation (369) shows that ϕ is zero and so

$$\mathbf{F} = \nabla \times \mathbf{A}.$$

But if the vector is defined only in a limited region, we must first extend it outside the region so as to obtain a vector solenoidal everywhere. To do this let F_n be the normal component of $\mathbf{F}$ on the boundary and ϕ' the function harmonic outside the region and having at points of the boundary a normal derivative equal to $-F_n$. At outside points take $\mathbf{F} = -\nabla\phi'$. The vector thus defined is solenoidal everywhere, since it is solenoidal inside the region and outside and has a normal component continuous on the boundary (§50). It is also regular at infinity. For it the function ϕ is therefore zero.

PROBLEMS

1. Inside a sphere of radius b a function has the value

$$\phi = b^2 - r^2,$$

where r is the distance from the origin. Outside the sphere it is zero. Find the distribution of charges having this function as potential.

2. Inside a sphere a function is

$$\phi = r^2,$$

where r is the distance from the origin. Outside the sphere it is zero. Find the distribution of charges having this potential.

3. A cube of edge b has three faces in the coördinate planes. Find the distribution of charges which has a potential

$$\phi = x^2$$

at points inside the cube and $\phi = 0$ at points outside.

4. At each point within a sphere of radius b and center at the origin, the electric intensity is

$$\mathbf{E} = \mathbf{R}$$

and outside the sphere it is zero. Find a distribution of charges which has this electric intensity.

5. At each point at distance $r < b$ from the z-axis the potential is $\phi = r^2$ and where r is greater than b

$$\phi = b^2\left(1 + \ln\frac{r}{b}\right)$$

Find the distribution which determines this potential.

6. At each point inside a sphere of radius b and center at the origin the electric intensity is $\mathbf{E} = 2i x$. Outside it is

$$\mathbf{E} = b^5 \nabla\left(\frac{x^2}{r^5} - \frac{1}{3r^3}\right),$$

where r is the distance from the center. Find a distribution which determines this electric intensity.

7. If $\mathbf{F}$ has the same direction at all points of the region V, show that

$$\mathbf{A} = \frac{1}{4\pi}\int_V \mathbf{F} \times \nabla\frac{1}{r}\,dv,$$

is perpendicular to that direction.

8. If $\mathbf{E}$ is the electric intensity due to a charge on a surface S, show that

$$\phi = -\frac{1}{4\pi}\int \mathbf{E} \cdot \nabla\frac{1}{r}\,dv,$$

taken over all space, is equal to the ordinary expression for its potential.

9. If $\mathbf{E}$ is the electric intensity due to a space charge in a finite region V, show that

$$\phi = -\frac{1}{4\pi}\int \mathbf{E} \cdot \nabla\frac{1}{r}\,dv,$$

taken over all space, is equal to the ordinary expression for the potential of such a charge.

10. If $\mathbf{E}$ is the electric intensity due to a double layer on a surface S, show that

$$\mathbf{A} = \frac{1}{4\pi}\int \mathbf{E} \times \nabla \frac{1}{r}\, dv,$$

taken over all space, is equal to the ordinary expression for vector potential of the double layer.

11. Two vectors $\mathbf{F}_1$, $\mathbf{F}_2$ defined in regions V_1, V_2 have potentials ϕ_1, $\mathbf{A}_1$ and ϕ_2, $\mathbf{A}_2$ respectively. If the regions V_1, V_2 have no interior points in common and

$$\phi = \phi_1 + \phi_2, \quad \mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2,$$

show that the vector

$$\mathbf{F} = -\nabla\phi + \nabla \times \mathbf{A}$$

has the value $\mathbf{F}_1$ at points of V_1 and $\mathbf{F}_2$ at points of V_2.

12. Inside a sphere of radius b and center at the origin a vector has the value

$$\mathbf{F} = \mathbf{j}x - \mathbf{i}y$$

and outside it is zero. Find the vector potential of $\mathbf{F}$ and show that its scalar potential is zero.

13. If ω is the solid angle subtended by a closed curve and

$$\mathbf{F} = -\nabla\omega,$$

express the vector potential of $\mathbf{F}$ as an integral around the curve.

14. Show that the components of

$$\mathrm{W} = \int_V \frac{\mathbf{F}\, dv}{r}$$

are harmonic in the region V if and only if $\mathbf{F}$ is zero in that region.

15. If $\mathbf{F}$ is continuously differentiable in the region V, show that

$$\phi = -\frac{1}{4\pi}\int_V \mathbf{F} \cdot \nabla \frac{1}{r}\, dv$$

is zero if and only if the vector which is equal to $\mathbf{F}$ at points of the region and zero at points outside, is solenoidal.

16. If $\mathbf{F}$ is continuously differentiable in the region V, show that

$$\mathbf{A} = \frac{1}{4\pi}\int_V \mathbf{F} \times \nabla \frac{1}{r}\, dv$$

is zero if and only if the vector which is equal to $\mathbf{F}$ at points of the region and zero at points outside, is irrotational.

17. If ϕ and its first derivatives are continuous in a region V of boundary S, show that

$$\frac{1}{4\pi}\int_V \nabla\phi \cdot \nabla \frac{1}{r}\, dv - \frac{1}{4\pi}\int_S \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) da$$

is equal to ϕ at points inside the region and zero at points outside.

18. If $\mathbf{F}$ and its first derivatives are continuous at all points of a region V of boundary S, show that

$$\frac{1}{4\pi}\int_V \left[(\nabla\cdot\mathbf{F})\,\nabla\frac{1}{r}+(\nabla\times\mathbf{F})\times\nabla\frac{1}{r}\right]dv-\frac{1}{4\pi}\int_S\left[\mathbf{F}\left(\mathbf{n}\cdot\nabla\frac{1}{r}\right)-\mathbf{F}\times\left(\mathbf{n}\times\nabla\frac{1}{r}\right)\right]da$$

is equal to $\mathbf{F}$ at points inside the region and zero at points outside.

19. If $\mathbf{F}$ is piecewise differentiable everywhere, regular at infinity, and ϕ, $\mathbf{A}$ are its potentials, show that

$$\int \mathbf{F}^2\,dv=\int[(\nabla\phi)^2+(\nabla\times\mathbf{A})^2]\,dv,$$

the integrals being taken over all space.

20. Under the conditions of the preceding problem show that

$$\frac{1}{4\pi}\int\mathbf{F}^2\,dv=\int(\phi\rho+\mathbf{A}\cdot\mathbf{I})\,dv+\int(\phi\sigma+\mathbf{A}\cdot\mathbf{J})da,$$

the integrals on the right being taken over all values of their integrands.

CHAPTER IX

RETARDED POTENTIALS

90. Variable Fields. In the fields so far discussed the quantities under consideration have been functions of position but not of the time. If a quantity f also varies with the time let

$$f(x, y, z, t)$$

be its value at the point (x, y, z) at time t. Since x, y, z, t are independent variables, the totality of such values constitutes a four-dimensional field.

There are two ways of selecting a three-dimensional set of values, or cross-section, of such a field. First we may associate together values which exist at a given instant of time. This simultaneous set of values constitutes a field of the kind we have been discussing. At each instant there is such a field and the problem, for example of hydrodynamics, is to determine how the field at one instant is related to those existing at earlier or later times.

In some cases, for example in the electromagnetic field, an occurrence at a point $Q(\xi, \eta, \zeta)$ does not produce immediate effects at a distant point $P(x, y, z)$, but only during a later interval when its influence, traveling with constant velocity c, is passing P. The state of the field at the point P at time t is determined by states existing at points Q at the earlier times

$$t' = t - \frac{r}{c},$$

where $r = PQ$. In such cases we associate the value $f(x, y, z, t)$ at P at time t with the values

$$f\left(\xi, \eta, \zeta, t - \frac{r}{c}\right)$$

existing at points $Q(\xi, \eta, \zeta)$ at times $t - \frac{r}{c}$. *These values constitute the field as seen at time t by an observer at P using light of*

velocity c. The result is a cross-section of the four-dimensional field which depends not only on the time but also on the position of P.

91. Expression of a Scalar Function as Retarded Potential. Let $\phi(x, y, z, t)$ be a scalar function, which for the values considered has continuous second derivatives with respect to x, y, z, t, and let r be the distance between the points $P(x, y, z)$ and $Q(\xi, \eta, \zeta)$. In the following discussions we shall use the notation

$$\phi = \phi(x, y, z, t)$$

to represent the value at P at time t and

$$\phi' = \phi(\xi, \eta, \zeta, t')$$

to represent the value at Q at time

$$t' = t - \frac{r}{c}. \tag{385}$$

Likewise

$$\nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} \tag{386}$$

is the gradient at P obtained by differentiating ϕ with t constant and

$$\nabla\phi' = \mathbf{i}\frac{\partial\phi'}{\partial\xi} + \mathbf{j}\frac{\partial\phi'}{\partial\eta} + \mathbf{k}\frac{\partial\phi'}{\partial\zeta} \tag{387}$$

is the gradient at Q obtained by differentiating ϕ' with t' constant. In some cases we shall differentiate with respect to the coördinates of Q with t constant. We represent this operation by the notation ∇_t.

To obtain expressions analogous to those of §82 we apply the divergence theorem to the vector

$$\mathbf{U}(x, y, z, \xi, \eta, \zeta, t) = \frac{1}{r}\nabla\phi' - \left(\phi' + \frac{r}{c}\frac{\partial\phi'}{\partial t'}\right)\nabla\frac{1}{r}, \tag{388}$$

where

$$\nabla\frac{1}{r} = \nabla_Q\frac{1}{r}$$

is the gradient at Q. For this we shall need the divergence of $\mathbf{U}$ with respect to ξ, η, ζ with t constant. In the differentiations

ϕ' is a function of ξ, η, ζ, t' and t' is the function of ξ, η, ζ, t determined by (385). By direct operation we obtain

$$\nabla_t \cdot \mathbf{U} = \frac{1}{r}\nabla^2\phi' - \frac{1}{rc}\nabla\frac{\partial\phi'}{\partial t'}\cdot\nabla r + \nabla\frac{1}{r}\cdot\nabla\phi'$$
$$-\nabla\frac{1}{r}\cdot\left[\nabla\phi' - \frac{1}{c}\frac{\partial\phi'}{\partial t'}\nabla r + \frac{1}{c}\frac{\partial\phi'}{\partial t'}\nabla r + \frac{r}{c}\nabla\frac{\partial\phi'}{\partial t'} - \frac{r}{c^2}\frac{\partial^2\phi'}{\partial t'^2}\nabla r\right]$$

which after cancellations and reductions becomes

$$\nabla_t \cdot \mathbf{U} = \frac{1}{r}\left(\nabla^2\phi' - \frac{1}{c^2}\frac{\partial^2\phi'}{\partial t'^2}\right). \tag{389}$$

Let $\rho(x, y, z, t)$ be the function defined by the equation

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho \tag{390}$$

and

$$\rho' = \rho(\xi, \eta, \zeta, t')$$

its value at $Q(\xi, \eta, \zeta)$ at time t'. Equation (389) is then equivalent to

$$\nabla_t \cdot \mathbf{U} = -\frac{4\pi\rho'}{r}.$$

Let S_0 be a small sphere of center P and S the boundary of the region V under consideration. The divergence theorem gives

$$\int_S \mathbf{U}\cdot\mathbf{n}\,da + \int_{S_0}\mathbf{U}\cdot\mathbf{n}\,da = -4\pi\int_{V'}\frac{\rho'\,dv}{r}, \tag{391}$$

where V' is the region between S_0 and S. When the radius of S_0 approaches zero

$$\lim\int_{S_0}\mathbf{U}\cdot\mathbf{n}\,da = -4\pi\phi(x, y, z, t).$$

From (391) we therefore obtain

$$\phi = \int_V\frac{\rho'\,dv}{r} + \frac{1}{4\pi}\int_S\left[\frac{1}{r}\frac{\partial\phi'}{\partial n} - \left(\phi' + \frac{r}{c}\frac{\partial\phi'}{\partial t'}\right)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]da. \tag{392}$$

If the region V contains a surface S_1 on which the normal derivative of ϕ has a discontinuity

$$\frac{\partial\phi}{\partial n_+} - \frac{\partial\phi}{\partial n_-} = -4\pi\sigma,$$

or a surface S_2 on which ϕ itself has a discontinuity

$$\phi_+ - \phi_- = 4\pi\tau,$$

as in §82 the above equation is replaced by

$$\phi = \int_V \frac{\rho'\,dv}{r} + \int_{S_1} \frac{\sigma'\,da}{r} + \int_{S_2} \left(\tau' + \frac{r}{c}\frac{\partial\tau'}{\partial t'}\right)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)da$$
$$+ \frac{1}{4\pi}\int_S \left[\frac{1}{r}\frac{\partial\phi'}{\partial n} - \left(\phi' + \frac{r}{c}\frac{\partial\phi'}{\partial t'}\right)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]da.$$

These integrals, expressing the value of a function at time t in terms of boundary values and characteristics of the function at time t', are called **retarded potentials.** In electromagnetic theory ρ', σ', τ' represent space charge, surface charge, and moment of double layer at time t'. The above discussion shows that *any function of x, y, z, t with sufficient continuity is the retarded potential of a certain distribution of charges.*

92. Derivatives of Retarded Potentials. The retarded potential at $P(x, y, z)$ due to a space charge of density ρ spread over a region V is

$$\phi(x, y, z, t) = \int_V \frac{\rho'\,dv}{r}, \tag{393}$$

where r is the distance between P and the point $Q(\xi, \eta, \zeta)$ at which the element of integration is located, and

$$\rho' = \rho(\xi, \eta, \zeta, t')$$

is the density of charge at Q at time

$$t' = t - \frac{r}{c}.$$

We assume that ρ has continuous second derivatives with respect to the time. At points where ρ also has continuous first derivatives with respect to x, y, z, we shall show that ϕ has continuous second derivatives which satisfy the equation

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho. \tag{394}$$

To show this suppose these conditions are satisfied at P_0 at time t and let

$$\phi_0(x, y, z, t) = \int_V \frac{1}{r}\left(\rho_0 + \frac{r_0}{c}\frac{\partial\rho_0}{\partial t_0}\right)dv,$$

where r_0 is the distance between P_0 and the element of integration at $Q(\xi, \eta, \zeta)$,

$$\rho_0 = \rho(\xi, \eta, \zeta, t_0),$$

and

$$t_0 = t - \frac{r_0}{c}.$$

Since t_0 is not a function of x, y, z, this is a potential of the type discussed in Chapter VI. Hence $\nabla^2\phi_0$ is equal to -4π times the value of

$$\rho_0 + \frac{r_0}{c}\frac{\partial \rho_0}{\partial t_0}$$

at P_0, or

$$\nabla^2\phi_0 = -4\pi\rho(x_0, y_0, z_0, t). \tag{395}$$

Take P_0 as origin and let $P(x, y, z)$ be a point on the x-axis. By direct differentiation we obtain

$$\frac{\partial}{\partial x}(\phi - \phi_0) = \int_V \left(\rho' + \frac{r}{c}\frac{\partial \rho'}{\partial t'} - \rho_0 - \frac{r_0}{c}\frac{\partial \rho_0}{\partial t_0}\right)\frac{\partial}{\partial x}\left(\frac{1}{r}\right)dv.$$

Since

$$\rho' + \frac{r}{c}\frac{\partial \rho'}{\partial t'} - \rho_0 - \frac{r_0}{c}\frac{\partial \rho_0}{\partial t_0}$$

is zero when $P = P_0$, by the mean value theorem it is equal to x times the value of its derivative with respect to x, namely

$$-\frac{r}{c^2}\frac{\partial^2 \rho'}{\partial t'^2}\frac{\partial r}{\partial x},$$

at some point between P and P_0. Hence

$$\frac{\frac{\partial}{\partial x}(\phi - \phi_0)}{x} = -\int_V \left[\frac{r}{c^2}\frac{\partial^2 \rho'}{\partial t'^2}\frac{\partial r}{\partial x}\right]_m \frac{\partial}{\partial x}\left(\frac{1}{r}\right)dv,$$

where the subscript m signifies a mean value for the expression in brackets. When x approaches zero, this gives for the second derivative at P_0

$$\frac{\partial^2}{\partial x^2}(\phi - \phi_0) = \int_V \frac{(x - \xi)^2}{c^2 r^3}\frac{\partial^2 \rho'}{\partial t'^2}\,dv.$$

In a similar way we obtain

$$\frac{\partial^2}{\partial y^2}(\phi - \phi_0) = \int_V \frac{(y-\eta)^2}{c^2 r^3} \frac{\partial^2 \rho'}{\partial t'^2} dv,$$

$$\frac{\partial^2}{\partial z^2}(\phi - \phi_0) = \int_V \frac{(z-\zeta)^2}{c^2 r^3} \frac{\partial^2 \rho'}{\partial t'^2} dv,$$

and by addition

$$\nabla^2(\phi - \phi_0) = \int_V \frac{1}{c^2 r} \frac{\partial^2 \rho'}{\partial t'^2} dv.$$

Since

$$\frac{\partial^2 \rho'}{\partial t'^2} = \frac{\partial^2 \rho'}{\partial t^2},$$

from (393) we have

$$\frac{\partial^2 \phi}{\partial t^2} = \int_V \frac{1}{r} \frac{\partial^2 \rho'}{\partial t'^2} dv.$$

Hence

$$\nabla^2(\phi - \phi_0) = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}.$$

Transposing and using (395) this becomes

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -4\pi\rho$$

which was to be proved.

If

$$\phi = \int_S \frac{\sigma' \, da}{r} \tag{396}$$

is the retarded potential of a surface charge,

$$\nabla\phi = -\int_S \left(\sigma' + \frac{r}{c} \frac{\partial \sigma'}{\partial t'}\right) \nabla \frac{1}{r} da,$$

$$\nabla^2 \phi = \int_S \frac{1}{c^2 r} \frac{\partial^2 \sigma'}{\partial t'^2} da = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}.$$

This potential therefore satisfies the equation

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0. \tag{397}$$

The potential of a double layer can be written

$$\phi = \int_S \left(\tau' + \frac{r}{c}\frac{\partial \tau'}{\partial t'}\right)\frac{\partial}{\partial n}\left(\frac{1}{r}\right) da = -\nabla \cdot \mathbf{W},$$

where

$$\mathbf{W} = \int_S \frac{\tau' \mathbf{n}\, da}{r}.$$

As in the preceding case we show that $\mathbf{W}$, and so also ϕ, satisfies equation (397).

If σ has a continuous first derivative and τ a continuous second derivative with respect to the time, the expressions for $\nabla\phi$ in case of these surface distributions differ from those of Chapter VI by terms which are continuous in the neighborhood of the surface S. Under these conditions the retarded potentials and their gradients therefore have discontinuities of the same types as the steady potentials.

93. The Wave Equation. At a point where there is no charge each of the potentials just discussed satisfies the equation

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0. \tag{398}$$

In case of waves traveling with velocity c in a homogeneous medium, the displacement of a particle of the medium satisfies this equation. It is therefore called the **wave equation** and a function which satisfies it is often called a **wave function.**

For such a function equation (392) takes the form*

$$\phi = \frac{1}{4\pi}\int_S \left[\frac{1}{r}\frac{\partial \phi'}{\partial n} - \left(\phi' + \frac{r}{c}\frac{\partial \phi'}{\partial t'}\right)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] da. \tag{399}$$

The value at P at time t is thus determined by values and derivatives at points Q of a surrounding surface at times $t - \frac{r}{c}$. Each vector element $\mathbf{n}\, da$ of such a surface may be regarded as continuously emitting waves which travel with velocity c and when they reach a point P at distance r have the intensity

$$\left[\frac{1}{r}\frac{\partial \phi'}{\partial n} + \nabla_P \cdot \left(\frac{\mathbf{n}\phi'}{r}\right)\right] da,$$

* This is the mathematical formulation of what is known as Huygens' principle in optics.

where ∇_P signifies differentiation with respect to the coördinates of P. The value of ϕ at an interior point P at time t is the sum of the intensities of the waves then passing P.

If ϕ is a function of only one coördinate (for example x), or of two coördinates (x and y), equation (398) takes the simpler forms

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0, \tag{400}$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0, \tag{401}$$

representing the wave equation in one and two dimensions respectively. Each of these is related to a variable field in somewhat the same way that the corresponding Laplace's equation is related to a stationary field. If ct is plotted as a coördinate, (400), (401), and (398) differ only in one algebraic sign from Laplace's equation in two, three, and four dimensions respectively. We might then expect wave functions in any dimension to be similar to harmonic functions in the next higher dimension. In some ways they are, but in others they differ so radically as to make comparison of little value.

The principal reason for this difference is that *harmonic functions are analytic whereas wave functions need not be.* By this we mean that a function ϕ harmonic at a point P can be expressed as a power series in its variables convergent in a certain region about P. The coefficients in such a series are determined by values of ϕ as close to P as we please. In particular, if ϕ is zero in a small region about P, these coefficients are all zero and so ϕ is zero throughout a regular region in which it is harmonic. But it is possible for a wave function during an interval of time to be zero in a region about P, then have values while the wave is passing, and finally again be zero. Such behavior is quite impossible for a function analytic in the coördinates and the time.

For this reason the conditions by which wave functions are determined are different from those used for harmonic functions. The usual problem is, given a closed region V of boundary S and three continuous functions $f_1(x, y, z)$, $f_2(x, y, z)$, $f_3(x, y, z, t)$,

to find a wave function ϕ which at time t_0 satisfies the conditions

$$\phi = f_1, \qquad \frac{\partial \phi}{\partial t} = f_2 \tag{402}$$

at all points of V, and in the interval between t_0 and t satisfies the condition

$$\phi = f_3 \tag{403}$$

at all points of S. To show the existence of a function satisfying these conditions is a problem like the Dirichlet problem. In applications the existence of the function is usually proved by finding it. We shall merely show that these conditions determine the function, that if two wave functions have continuous second derivatives and satisfy these conditions, they are identical.

To show this let ϕ_1, ϕ_2 be wave functions which satisfy the conditions (402) and (403). The function

$$\phi = \phi_2 - \phi_1$$

is then a wave function which at time t_0 satisfies the equations

$$\phi = 0, \qquad \frac{\partial \phi}{dt} = 0 \tag{404}$$

at all points of V, and in the interval between t_0 and t satisfies the equation

$$\phi = 0 \tag{405}$$

at all points of S. Let

$$u = \int_V \left[(\nabla \phi)^2 + \frac{1}{c^2} \left(\frac{\partial \phi}{\partial t} \right)^2 \right] dv,$$

the integration being performed with t constant. This is a function of t with derivative

$$\frac{du}{dt} = 2 \int_V \left[\nabla \phi \cdot \nabla \frac{\partial \phi}{\partial t} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \frac{\partial^2 \phi}{\partial t^2} \right] dv.$$

From the wave equation we have

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi.$$

Hence

$$\begin{aligned} \frac{du}{dt} &= 2 \int_V \left[\nabla \phi \cdot \nabla \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial t} \nabla^2 \phi \right] dv \\ &= 2 \int_V \nabla \cdot \frac{\partial \phi}{\partial t} \nabla \phi \, dv = 2 \int_S \frac{\partial \phi}{\partial t} \mathbf{n} \cdot \nabla \phi \, da. \end{aligned}$$

Since ϕ is constant on S, the last integral is zero. Hence u is constant. But condition (404) shows that u is zero at time t_0. Consequently

$$u = 0.$$

Since the integrand is never negative

$$(\nabla\phi)^2 + \frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 = 0,$$

whence

$$\phi = \text{constant}.$$

But at time t_0, ϕ is zero. Therefore, finally

$$\phi = \phi_2 - \phi_1 = 0,$$

which was to be proved.

In this discussion it is not necessary that t be greater than t_0. The problem of finding a function which satisfies conditions (402) and (403) is formally the same whether t_0 is a future or a past time. Likewise in the retarded potentials the results we have obtained remain valid if the sign of c is changed. *The value of a potential at time t is thus related to future values in precisely the same way it is related to past.*

94. Expression of a Vector in Terms of Retarded Potentials. Let $\mathbf{F}(x, y, z, t)$ be a vector which in a region V has continuous second derivatives with respect to the time, and continuous or piecewise continuous first derivatives with respect to x, y, z. For generality we shall consider $\mathbf{F}$ as a complex quantity

$$\mathbf{F} = \mathbf{F}_1 + i\mathbf{F}_2,$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ are real vectors and

$$i = \sqrt{-1}.$$

The case where $\mathbf{F}$ is real is obtained by replacing $\mathbf{F}_2$ by zero.

With each point $P(x, y, z)$ of space associate the vector

$$\mathbf{W}(x, y, z, t) = \int_V \frac{\mathbf{F}'\, dv}{r}, \tag{406}$$

where $\mathbf{F}'$ is the value of $\mathbf{F}$ at the element of integration at time

$$t' = t - \frac{r}{c}.$$

Each component of $\mathbf{W}$ is a retarded potential of the type discussed in §92. At a point where $\mathbf{F}$ is differentiable we can therefore apply (394) to each component and so get

$$\nabla^2\mathbf{W} - \frac{1}{c^2}\frac{\partial^2\mathbf{W}}{\partial t^2} = -4\pi\mathbf{F}. \tag{407}$$

Let $\phi(x, y, z, t)$ be the scalar and $\mathbf{A}(x, y, z, t)$ the vector function defined by the equations

$$4\pi i\phi = \nabla \cdot \mathbf{W}, \tag{408}$$

$$4\pi\mathbf{A} = \nabla \times \mathbf{W} + \frac{i}{c}\frac{\partial\mathbf{W}}{\partial t}. \tag{409}$$

Since

$$\nabla \cdot \frac{\partial\mathbf{W}}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{W}),$$

these functions satisfy the equation

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0. \tag{410}$$

From (409) we have

$$4\pi\nabla \times \mathbf{A} = \nabla \times (\nabla \times \mathbf{W}) + \frac{i}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{W}),$$

$$\frac{i}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{W}) = \frac{4\pi i}{c}\frac{\partial\mathbf{A}}{\partial t} + \frac{1}{c^2}\frac{\partial^2\mathbf{W}}{\partial t^2}.$$

Adding the sum of these equations to

$$\nabla \times (\nabla \times \mathbf{W}) = \nabla(\nabla \cdot \mathbf{W}) - \nabla^2\mathbf{W}$$

and using (407) and (408), we can eliminate $\mathbf{W}$ and so obtain

$$\mathbf{F} = \nabla \times \mathbf{A} - i\left(\nabla\phi + \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right). \tag{411}$$

The quantities ϕ, $\mathbf{A}$ may be considered as a scalar and a vector potential associated with the vector $\mathbf{F}$. If $\mathbf{F}$ is piecewise continuous, these potentials are continuous at all points of space. They satisfy (411) at points of the region V where $\mathbf{F}$ has continuous first derivatives.

95. Expression of the Potentials in Terms of Characteristics of the Vector. In the closed region V let $\mathbf{F}(x, y, z, t)$ be con-

tinuously differentiable provided the boundary be regarded as extending on both sides of certain surfaces S_1. By differentiating with respect to the coördinates of $P(x, y, z)$ and making the usual change to coördinates at the element of integration, we obtain

$$\nabla \cdot \mathbf{W} = -\int_V \left(\mathbf{F}' + \frac{r}{c}\frac{\partial \mathbf{F}'}{\partial t'}\right) \cdot \nabla \frac{1}{r}\, dv,$$

$$\nabla \times \mathbf{W} = \int_V \left(\mathbf{F}' + \frac{r}{c}\frac{\partial \mathbf{F}'}{\partial t'}\right) \times \nabla \frac{1}{r}\, dv.$$

These expressions are equivalent to

$$\nabla \cdot \mathbf{W} = \int_V \left[\frac{\nabla \cdot \mathbf{F}'}{r} - \nabla_t \cdot \left(\frac{\mathbf{F}'}{r}\right)\right] dv,$$

$$\nabla \times \mathbf{W} = \int_V \left[\frac{\nabla \times \mathbf{F}'}{r} - \nabla_t \times \left(\frac{\mathbf{F}'}{r}\right)\right] dv,$$

where $\nabla \cdot \mathbf{F}'$, $\nabla \times \mathbf{F}'$ signify differentiation with t' constant and ∇_t signifies differentiation with t constant. From the divergence theorem and equation (125) we thus have

$$\nabla \cdot \mathbf{W} = \int_V \frac{\nabla \cdot \mathbf{F}'\, dv}{r} - \int \frac{\mathbf{F}' \cdot \mathbf{n}\, da}{r}, \tag{412}$$

$$\nabla \times \mathbf{W} = \int_V \frac{\nabla \times \mathbf{F}'\, dv}{r} - \int \frac{\mathbf{n} \times \mathbf{F}'\, da}{r}, \tag{413}$$

the surface integrals being extended over the outer boundary and both sides of the surfaces S_1 (cf. page 70).

Let $\Delta\mathbf{F}$ be the increment in $\mathbf{F}$ when the point $P(x, y, z)$ crosses one of the surfaces S_1 in the positive direction of the normal. Let ρ, $\mathbf{I}$, σ, $\mathbf{J}$ be the quantities defined by the equations

$$\nabla \cdot \mathbf{F} = 4\pi i\rho, \tag{414}$$

$$\nabla \times \mathbf{F} + \frac{i}{c}\frac{\partial \mathbf{F}}{\partial t} = 4\pi \mathbf{I}, \tag{415}$$

$$\mathbf{n} \cdot \Delta\mathbf{F} = 4\pi i\sigma, \tag{416}$$

$$\mathbf{n} \times \Delta\mathbf{F} = 4\pi \mathbf{J}. \tag{417}$$

Evaluating the integrals (412), (413) and inserting these values we obtain

$$\phi = \int_V \frac{\rho'\,dv}{r} + \int_{S_1} \frac{\sigma'\,da}{r} + \frac{i}{4\pi}\int_S \frac{\mathbf{F}'\cdot\mathbf{n}\,da}{r}, \tag{418}$$

$$\mathbf{A} = \int_V \frac{\mathbf{I}'\,dv}{r} + \int_{S_1} \frac{\mathbf{J}'\,da}{r} - \frac{1}{4\pi}\int_S \frac{\mathbf{n}\times\mathbf{F}'\,da}{r}, \tag{419}$$

where S is the surface forming the boundary of V.

If the region extends to infinity and $\mathbf{F}$ is regular at infinity, the boundary integrals approach zero when S increases indefinitely and so

$$\phi = \int \frac{\rho'\,dv}{r} + \int \frac{\sigma'\,da}{r}, \tag{420}$$

$$\mathbf{A} = \int \frac{\mathbf{I}'\,dv}{r} + \int \frac{\mathbf{J}'\,da}{r}, \tag{421}$$

the integrals being taken over all values of ρ', σ', $\mathbf{I}'$, $\mathbf{J}'$. The preceding equations are merely these equations for the vector which equals $\mathbf{F}$ at points within the region V and is zero outside.

96. Equation of Continuity. The quantities ρ, σ, $\mathbf{I}$, $\mathbf{J}$ are not arbitrary. They are subject to the equation of continuity (410), namely,

$$\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0. \tag{422}$$

The effect of this condition is to make ρ, σ, $\mathbf{I}$, $\mathbf{J}$ *behave like densities and currents of something which is nowhere created or destroyed.* In the electromagnetic field they represent densities and currents of electricity.

We assume ρ, σ, $\mathbf{I}$, $\mathbf{J}$ are piecewise differentiable. Proceeding as in §87 we obtain

$$\nabla\cdot\mathbf{A} = \int \frac{\nabla\cdot\mathbf{I}'\,dv}{r} + \int \frac{(\mathbf{n}\cdot\Delta\mathbf{I}' + \nabla\cdot\mathbf{J}')\,da}{r} + \int \frac{\mathbf{n}\times\Delta\mathbf{J}'\cdot d\mathbf{R}}{r},$$

the integrals being extended over all values of the various quantities. Also

$$\frac{1}{c}\frac{\partial\phi}{\partial t} = \int \frac{1}{cr}\frac{\partial\rho'}{\partial t'}\,dv + \int \frac{1}{cr}\frac{\partial\sigma'}{\partial t'}\,da.$$

If the sum of these expressions is zero for all positions of P, we must have

$$\nabla \cdot \mathbf{I} + \frac{1}{c}\frac{\partial \rho}{\partial t} = 0, \tag{423}$$

$$\mathbf{n} \cdot \Delta\mathbf{I} + \nabla \cdot \mathbf{J} + \frac{1}{c}\frac{\partial \sigma}{\partial t} = 0, \tag{424}$$

$$\mathbf{n} \times \Delta\mathbf{J} \cdot d\mathbf{R} = 0. \tag{425}$$

Considering ρ, σ, $\mathbf{I}$, $\mathbf{J}$ as density and current of something, equation (423) expresses the continuity of this material at a point in space, that the amount which flows away from a point is equal to the loss in the local supply. Equations (424) and (425) express the same fact at points where the space or surface currents are discontinuous. The above discussion shows that equation (422) is the necessary and sufficient condition that ρ, σ, $\mathbf{I}$, $\mathbf{J}$ have this conservative property.

97. The Electromagnetic Field. In a region which does not contain dielectrics or magnetic material, the electromagnetic field is characterized by the electric intensity $\mathbf{E}$ and the magnetic force $\mathbf{H}$. These satisfy Faraday's law of electromagnetic induction, expressed by the equation

$$\oint \mathbf{E} \cdot d\mathbf{R} = -\frac{1}{c}\frac{\partial}{\partial t}\int \mathbf{H} \cdot \mathbf{n}\, da. \tag{426}$$

The integral on the right is taken over any area and that on the left around its boundary, the directions of $\mathbf{n}$ and $d\mathbf{R}$ being related as in Stokes's theorem. For this equation to hold it is necessary that the surface integral depend only on the boundary and not otherwise on the area over which it is taken. Consequently

$$\frac{\partial}{\partial t}\oint \mathbf{H} \cdot \mathbf{n}\, da = 0,$$

and so

$$\oint \mathbf{H} \cdot \mathbf{n}\, da = c$$

may depend on the closed surface over which the integral is taken but not on the time. If then there is any process, consistent with Faraday's law, by which the field could be created or reduced to zero, the constant is zero. This is usually assumed to be the case, that is,

$$\oint \mathbf{H} \cdot \mathbf{n}\, da = 0. \tag{427}$$

At points where $\mathbf{H}$ and $\mathbf{E}$ are continuously differentiable, equations (426) and (427) require

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = 0, \tag{428}$$

$$\nabla \cdot \mathbf{H} = 0. \tag{429}$$

On a surface of discontinuity, if $\mathbf{E}$, $\mathbf{H}$, and $\frac{\partial \mathbf{H}}{\partial t}$ are bounded, they also require that the tangential component of $\mathbf{E}$ and the normal component of $\mathbf{H}$ be continuous, that is,

$$\mathbf{n} \times \Delta \mathbf{E} = 0, \tag{430}$$

$$\mathbf{n} \cdot \Delta \mathbf{H} = 0. \tag{431}$$

In the general equations of §§94 and 95, let

$$\mathbf{F} = \mathbf{H} + i\mathbf{E}. \tag{432}$$

Since $\mathbf{H}$ and $\mathbf{E}$ are real, the above equations and (414) to (417) show that ρ, $\mathbf{I}$, σ, $\mathbf{J}$ are real. Suppose $\mathbf{H}$ and $\mathbf{E}$ are regular at infinity. When the region increases indefinitely the boundary integrals in the expressions for ϕ and $\mathbf{A}$ then approach zero. Hence ϕ and $\mathbf{A}$ are real. The equation

$$\mathbf{F} = \mathbf{H} + i\mathbf{E} = \nabla \times \mathbf{A} - i\left(\nabla\phi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right)$$

therefore gives

$$\mathbf{H} = \nabla \times \mathbf{A}, \tag{433}$$

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{434}$$

These and equations (420) to (431) are the usual equations of electromagnetic theory. It follows that *if any two vectors* $\mathbf{E}$ *and* $\mathbf{H}$ *satisfy Faraday's law the quantities* ρ, σ, $\mathbf{I}$, $\mathbf{J}$, ϕ, $\mathbf{A}$ *determined by equations* (414) *to* (421) *have the mathematical properties of charges, currents, and potentials in the electromagnetic field.*

CHAPTER X

LINEAR VECTOR FUNCTIONS

98. Linear Vector Functions. A vector

$$\mathbf{F}' = \mathbf{i}u' + \mathbf{j}v' + \mathbf{k}w'$$

is called a function of a vector

$$\mathbf{F} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$$

if the components of $\mathbf{F}'$ are functions of the components of $\mathbf{F}$. In particular, $\mathbf{F}'$ is called a **linear vector function** if its components are homogeneous linear functions of those of $\mathbf{F}$, that is, if

$$\begin{aligned} u' &= a_1u + b_1v + c_1w, \\ v' &= a_2u + b_2v + c_2w, \\ w' &= a_3u + b_3v + c_3w, \end{aligned}$$

where the a's, b's, c's are constant, or at least do not vary with u, v, w. In that case $\mathbf{F}'$ can be written

$$\mathbf{F}' = u\mathbf{A} + v\mathbf{B} + w\mathbf{C}, \tag{435}$$

where

$$\begin{aligned} \mathbf{A} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \\ \mathbf{B} &= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \\ \mathbf{C} &= c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}. \end{aligned}$$

Equation (435) is equivalent to

$$\mathbf{F}' = (\mathbf{F} \cdot \mathbf{i})\mathbf{A} + (\mathbf{F} \cdot \mathbf{j})\mathbf{B} + (\mathbf{F} \cdot \mathbf{k})\mathbf{C}. \tag{436}$$

This can be regarded as a product

$$\mathbf{F}' = \mathbf{F} \cdot \Phi$$

of $\mathbf{F}$ and an operator

$$\Phi = \mathbf{i}\mathbf{A} + \mathbf{j}\mathbf{B} + \mathbf{k}\mathbf{C},$$

the multiplication consisting in forming the product of $\mathbf{F}$ with the left factor in each term of Φ and adding the results. The terms $\mathbf{iA}$, $\mathbf{jB}$, $\mathbf{kC}$ are called dyads and Φ is called a dyadic.

99. Dyadics. If $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n, \mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n$ are vectors, the notation

$$\Phi = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \cdots + \mathbf{A}_n\mathbf{B}_n \tag{437}$$

is used to represent the operator (called a **dyadic**) which with a vector $\mathbf{F}$ forms the scalar products

$$\mathbf{F} \cdot \Phi = (\mathbf{F} \cdot \mathbf{A}_1)\mathbf{B}_1 + (\mathbf{F} \cdot \mathbf{A}_2)\mathbf{B}_2 + \cdots + (\mathbf{F} \cdot \mathbf{A}_n)\mathbf{B}_n,$$

$$\Phi \cdot \mathbf{F} = \mathbf{A}_1(\mathbf{B}_1 \cdot \mathbf{F}) + \mathbf{A}_2(\mathbf{B}_2 \cdot \mathbf{F}) + \cdots + \mathbf{A}_n(\mathbf{B}_n \cdot \mathbf{F}).$$

In the first of these $\mathbf{F}$ is called *prefactor*, in the second *postfactor*. The vectors $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ are called *antecedents*, the vectors $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n$ are called *consequents*. A dyadic

$$\Phi = \mathbf{AB},$$

consisting of a single term, is called a **dyad**.

Two dyadics Φ, Ψ are called equal if

$$\mathbf{F} \cdot \Phi = \mathbf{F} \cdot \Psi,$$

$$\Phi \cdot \mathbf{F} = \Psi \cdot \mathbf{F},$$

for every vector $\mathbf{F}$. A dyadic Φ is said to be zero if

$$\mathbf{F} \cdot \Phi = \Phi \cdot \mathbf{F} = 0,$$

for all values of $\mathbf{F}$. The distributive law,

$$(\mathbf{A} + \mathbf{B})(\mathbf{C} + \mathbf{D}) = \mathbf{AC} + \mathbf{AD} + \mathbf{BC} + \mathbf{BD}, \tag{438}$$

follows since equal results are obtained when both sides are multiplied by any vector. Also

$$(\lambda\mathbf{A})\mathbf{B} = \mathbf{A}(\lambda\mathbf{B}) = \lambda\mathbf{AB}, \tag{439}$$

where λ is any scalar, and

$$\lambda\mathbf{AB} = 0, \quad \text{if } \lambda = 0. \tag{440}$$

By expressing the antecedents in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, expanding, and collecting terms, any dyadic can be reduced to the form

$$\Phi = \mathbf{iA} + \mathbf{jB} + \mathbf{kC}. \tag{441}$$

In a similar way, by expressing the consequents in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, the dyadic can be reduced to the form

$$\Phi = \mathbf{A}'\mathbf{i} + \mathbf{B}'\mathbf{j} + \mathbf{C}'\mathbf{k}. \tag{442}$$

For a given dyadic the vectors **A**, **B**, **C** in (441) and **A′**, **B′**, **C′** in (442) are usually not the same.

From (441) we get

$$\mathbf{i} \cdot \Phi = \mathbf{A},$$
$$\mathbf{j} \cdot \Phi = \mathbf{B},$$
$$\mathbf{k} \cdot \Phi = \mathbf{C}.$$

The dyadic will thus be zero when and only when **A**, **B**, **C** are zero. *A dyadic which is zero can thus be reduced to a form which vanishes identically.* To make this reduction the only equations needed are (438), (439), and (440). Since the dot and cross product also satisfy these equations, it follows that *a linear equation satisfied by dyads will remain valid if each dyad is replaced by the dot or cross product of its two vectors.*

In addition to (438), (439), (440), other products, such as the dot and cross, have an equation

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A},$$
$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A},$$

connecting products taken in different orders. It is this equation which determines the character of the multiplication. Unless **A** and **B** are parallel, there is no equation connecting the dyads **AB**, **BA**. For this reason the dyad product is sometimes called the **indeterminate product.**

100. Conjugate Dyadics. The dyadic

$$\Phi_c = \mathbf{A}\mathbf{i} + \mathbf{B}\mathbf{j} + \mathbf{C}\mathbf{k},$$

obtained by interchanging the antecedent and consequent in each term of

$$\Phi = \mathbf{i}\mathbf{A} + \mathbf{j}\mathbf{B} + \mathbf{k}\mathbf{C},$$

is called the **conjugate** of Φ. If a dyadic is equal to its conjugate it is called self-conjugate, or **symmetric.** A dyadic equal to the negative of its conjugate is called anti-self-conjugate, or **anti-symmetric.**

The conjugate of a sum of dyadics is evidently equal to the sum of their conjugates. The dyadic

$$\Phi + \Phi_c$$

is therefore symmetric, and

$$\Phi - \Phi_c$$

is anti-symmetric. The equation

$$\Phi = \tfrac{1}{2}(\Phi + \Phi_c) + \tfrac{1}{2}(\Phi - \Phi_c)$$

shows that any dyadic can be expressed as the sum of a symmetric and an anti-symmetric dyadic.

The coördinate axes can be so chosen that a given symmetric dyadic takes the form

$$\Phi = a\mathbf{ii} + b\mathbf{jj} + c\mathbf{kk}, \tag{443}$$

where a, b, c are scalars.

To show this consider the expression

$$\mathbf{R} \cdot \Phi \cdot \mathbf{R},$$

where $\mathbf{R}$ is the vector from the origin to (x, y, z). Since this is a homogeneous quadratic in x, y, z a set of principal axes can be found such that it takes the form

$$\mathbf{R} \cdot \Phi \cdot \mathbf{R} = ax^2 + by^2 + cz^2.$$

If

$$\Psi = \Phi - a\mathbf{ii} - b\mathbf{jj} - c\mathbf{kk},$$

this equation is equivalent to

$$\mathbf{R} \cdot \Psi \cdot \mathbf{R} = 0.$$

This equation holds for all values of $\mathbf{R}$. Replacing $\mathbf{R}$ by $\mathbf{R}_1 + \mathbf{R}_2$ we get

$$(\mathbf{R}_1 + \mathbf{R}_2) \cdot \Psi \cdot (\mathbf{R}_1 + \mathbf{R}_2) = 0.$$

Expanding and using the equations

$$\mathbf{R}_1 \cdot \Psi \cdot \mathbf{R}_1 = \mathbf{R}_2 \cdot \Psi \cdot \mathbf{R}_2 = 0$$

we have

$$\mathbf{R}_1 \cdot \Psi \cdot \mathbf{R}_2 + \mathbf{R}_2 \cdot \Psi \cdot \mathbf{R}_1 = 0. \tag{444}$$

Since Φ is symmetric, Ψ is also symmetric. Hence

$$\mathbf{R}_1 \cdot \Psi \cdot \mathbf{R}_2 = \mathbf{R}_2 \cdot \Psi \cdot \mathbf{R}_1$$

and (444) is equivalent to

$$\mathbf{R}_1 \cdot \Psi \cdot \mathbf{R}_2 = 0.$$

Since this holds for all values of $\mathbf{R}_1$ and $\mathbf{R}_2$,

$$\mathbf{R}_1 \cdot \Psi = \Psi \cdot \mathbf{R}_2 = 0.$$

Hence Ψ is zero and so

$$\Phi = a\mathbf{ii} + b\mathbf{jj} + c\mathbf{kk},$$

which was to be proved.

Axes for which a symmetric dyadic takes the form (443) are called **principal axes** of the dyadic.

101. Scalar and Vector of a Dyadic. If in a given dyadic

$$\Phi = \mathbf{iA} + \mathbf{jB} + \mathbf{kC}$$

we replace each dyad by the scalar product of its two vectors, we obtain a quantity

$$\Phi_s = \mathbf{i} \cdot \mathbf{A} + \mathbf{j} \cdot \mathbf{B} + \mathbf{k} \cdot \mathbf{C} \tag{445}$$

called the **scalar of the dyadic.** If we replace each dyad by the cross product of its factors, we obtain a quantity

$$\Psi_v = \mathbf{i} \times \mathbf{A} + \mathbf{j} \times \mathbf{B} + \mathbf{k} \times \mathbf{C} \tag{446}$$

called the **vector of the dyadic.** In §99 we found that a linear equation satisfied by dyads remains valid if each dyad is replaced by the dot or cross products of its factors. From this it follows that the scalar and vector of a dyadic have the same values in whatever form the dyadic is expressed.

If Φ is anti-symmetric and $\mathbf{F}$ is any vector

$$\mathbf{F} \cdot \Phi = \tfrac{1}{2}\Phi_v \times \mathbf{F}.$$

To prove this let

$$\Phi = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \mathbf{A}_3\mathbf{B}_3.$$

Since Φ is anti-symmetric it can also be written

$$\Phi = \tfrac{1}{2}(\Phi - \Phi_c) = \tfrac{1}{2}(\mathbf{A}_1\mathbf{B}_1 - \mathbf{B}_1\mathbf{A}_1 + \mathbf{A}_2\mathbf{B}_2 - \mathbf{B}_2\mathbf{A}_2 + \mathbf{A}_3\mathbf{B}_3 - \mathbf{B}_3\mathbf{A}_3,$$

whence

$$\mathbf{F} \cdot \Phi = \tfrac{1}{2}[(\mathbf{F} \cdot \mathbf{A}_1)\mathbf{B}_1 - (\mathbf{F} \cdot \mathbf{B}_1)\mathbf{A}_1 + (\mathbf{F} \cdot \mathbf{A}_2)\mathbf{B}_2 - (\mathbf{F} \cdot \mathbf{B}_2)\mathbf{A}_2 + (\mathbf{F} \cdot \mathbf{A}_3)\mathbf{B}_3 - (\mathbf{F} \cdot \mathbf{B}_3)\mathbf{A}_3] = \tfrac{1}{2}[\mathbf{A}_1 \times \mathbf{B}_1 + \mathbf{A}_2 \times \mathbf{B}_2 + \mathbf{A}_3 \times \mathbf{B}_3] \times \mathbf{F},$$

which was to be proved.

The vanishing of its vector is the condition that a dyadic be symmetric.

To show this let

$$\Phi = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \mathbf{A}_3\mathbf{B}_3.$$

For Φ to be symmetric it is necessary and sufficient that

$$\mathbf{F} \cdot \Phi = \mathbf{F} \cdot \Phi_c$$

for all values of $\mathbf{F}$. Proceeding as above we show that this is equivalent to

$$[\mathbf{A}_1 \times \mathbf{B}_1 + \mathbf{A}_2 \times \mathbf{B}_2 + \mathbf{A}_3 \times \mathbf{B}_3] \times \mathbf{F} = 0,$$

and this will hold for all values of $\mathbf{F}$ if and only if

$$\Phi_v = \mathbf{A}_1 \times \mathbf{B}_1 + \mathbf{A}_2 \times \mathbf{B}_2 + \mathbf{A}_3 \times \mathbf{B}_3 = 0.$$

102. The Idemfactor. By direct multiplication of the vector

$$\mathbf{F} = \mathbf{i}f_1 + \mathbf{j}f_2 + \mathbf{k}f_3$$

and the dyadic

$$\mathbf{I} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} \tag{447}$$

we obtain

$$\mathbf{F} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{F} = \mathbf{F}.$$

Whether used as prefactor or postfactor multiplication by $\mathbf{I}$ thus leaves a vector unchanged. For this reason $\mathbf{I}$ is called the **idemfactor**, or identical dyadic.

The equation

$$\mathbf{A}'\mathbf{A} + \mathbf{B}'\mathbf{B} + \mathbf{C}'\mathbf{C} = \mathbf{I} \tag{448}$$

is the necessary and sufficient condition that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *and* $\mathbf{A}', \mathbf{B}', \mathbf{C}'$ *be reciprocal sets of vectors.*

First suppose $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{A}', \mathbf{B}', \mathbf{C}'$ are reciprocal sets (cf. §36). Consider the difference

$$\Psi = \mathbf{A}'\mathbf{A} + \mathbf{B}'\mathbf{B} + \mathbf{C}'\mathbf{C} - \mathbf{I}.$$

By direct multiplication and use of the relations

$$\mathbf{A} \cdot \mathbf{A}' = 1, \quad \mathbf{A} \cdot \mathbf{B}' = \mathbf{A} \cdot \mathbf{C}' = 0, \quad \mathbf{A} \cdot \mathbf{I} = \mathbf{A},$$

we get

$$\mathbf{A} \cdot \Psi = 0.$$

In a similar way we show that

$$\mathbf{B} \cdot \Psi = \mathbf{C} \cdot \Psi = 0.$$

Since **A, B, C** are linearly independent

$$\Psi = 0,$$

which was to be proved.

Conversely, if equation (448) is satisfied, the vectors **A, B, C** and **A′, B′, C′** form reciprocal sets. In the first place **A, B, C** are linearly independent. For if not there would be a vector **n** perpendicular to all three. Multiplying both sides of (448) by the postfactor **n** we should have

$$0 = \mathbf{n},$$

which is not correct. There is then a set of vectors $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{C}_1$ reciprocal to **A, B, C** (§36). If we multiply both sides of (448) by the postfactor $\mathbf{A}_1$, we obtain

$$\mathbf{A}' = \mathbf{A}_1.$$

In a similar way we prove

$$\mathbf{B}' = \mathbf{B}_1, \quad \mathbf{C}' = \mathbf{C}_1.$$

Consequently **A′, B′, C′** is the set reciprocal to **A, B, C.**

103. Product of Dyadics. The direct, or scalar, product of two dyads **AB** and **CD** is defined as the dyad

$$\mathbf{AB} \cdot \mathbf{CD} = (\mathbf{B} \cdot \mathbf{C})\mathbf{AD},$$

obtained by combining the adjacent factors **B, C** and leaving the other factors unchanged. The product of two dyadics

$$\Phi = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \mathbf{A}_3\mathbf{B}_3,$$
$$\Psi = \mathbf{C}_1\mathbf{D}_1 + \mathbf{C}_2\mathbf{D}_2 + \mathbf{C}_3\mathbf{D}_3,$$

is the sum of terms

$$\Phi \cdot \Psi = \mathbf{A}_1\mathbf{B}_1 \cdot \mathbf{C}_1\mathbf{D}_1 + \mathbf{A}_1\mathbf{B}_1 \cdot \mathbf{C}_2\mathbf{D}_2 + \cdots$$

obtained by multiplying the dyads distributively.

The product of two dyadics is thus a dyadic. Likewise the product of any number of dyadics taken in a definite order is a dyadic. It is further evident that these products satisfy the distributive law

$$(\Phi_1 + \Phi_2) \cdot (\Psi_1 + \Psi_2) = \Phi_1 \cdot \Psi_1 + \Phi_1 \cdot \Psi_2 + \Phi_2 \cdot \Psi_1 + \Phi_2 \cdot \Psi_2,$$

and the associative law

$$\Phi_1 \cdot (\Phi_2 \cdot \Phi_3) = (\Phi_1 \cdot \Phi_2) \cdot \Phi_3.$$

In some ways dyadics are thus more like the numbers of algebra than vectors are.

If $\mathbf{I}$ is the idemfactor,

$$\mathbf{AB} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{AB} = \mathbf{AB}.$$

Similarly, for the product of $\mathbf{I}$ and a dyadic we find

$$\mathbf{I} \cdot \Phi = \Phi \cdot \mathbf{I} = \Phi. \tag{449}$$

When multiplied by the idemfactor a dyad or dyadic is thus unchanged.

Two dyadics Φ, Ψ are called **reciprocal** if their product is equal to the identical dyadic, that is, if

$$\Psi \cdot \Phi = \mathbf{I} \tag{450}$$

or

$$\Phi \cdot \Psi = \mathbf{I}. \tag{451}$$

If either of these equations is satisfied the other is also. To show this suppose (450) is satisfied. Let

$$\Phi = \mathbf{iA} + \mathbf{jB} + \mathbf{kC},$$
$$\Psi = \mathbf{A'i} + \mathbf{B'j} + \mathbf{C'k}.$$

From (450) we then have

$$\mathbf{A'A} + \mathbf{B'B} + \mathbf{C'C} = \mathbf{I}.$$

Hence $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{A'}$, $\mathbf{B'}$, $\mathbf{C'}$ are reciprocal sets of vectors. Multiplying in the opposite order and using the equations connecting reciprocal sets, we get

$$\Phi \cdot \Psi = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} = \mathbf{I}.$$

The reciprocal of Φ is often represented by the notation Φ^{-1}. If

$$\Phi = \mathbf{iA} + \mathbf{jB} + \mathbf{kC}$$

and **A**, **B**, **C** are linearly independent, the above discussion shows that

$$\Phi^{-1} = \mathbf{A'i} + \mathbf{B'j} + \mathbf{C'k},$$

where **A'**, **B'**, **C'** is the set of vectors reciprocal to **A**, **B**, **C**. If **A**, **B**, **C** are not linearly independent Φ has no reciprocal.

104. Derivatives of a Vector. The partial derivatives

$$\frac{\partial \mathbf{F}}{\partial x}, \quad \frac{\partial \mathbf{F}}{\partial y}, \quad \frac{\partial \mathbf{F}}{\partial z}$$

of a vector function $\mathbf{F}(x, y, z)$ are vectors. These can be obtained by taking the scalar product of the dyadic

$$\nabla \mathbf{F} = \mathbf{i}\frac{\partial \mathbf{F}}{\partial x} + \mathbf{j}\frac{\partial \mathbf{F}}{\partial y} + \mathbf{k}\frac{\partial \mathbf{F}}{\partial z} \tag{452}$$

and the prefactors **i**, **j**, **k**. More generally, the derivative of **F** in the direction of the unit vector

$$\mathbf{u} = \frac{d\mathbf{R}}{ds} = \mathbf{i}\frac{dx}{ds} + \mathbf{j}\frac{dy}{ds} + \mathbf{k}\frac{dz}{ds}$$

is the product

$$\mathbf{u} \cdot \nabla \mathbf{F} = \frac{\partial \mathbf{F}}{\partial x}\frac{dx}{ds} + \frac{\partial \mathbf{F}}{\partial y}\frac{dy}{ds} + \frac{\partial \mathbf{F}}{\partial z}\frac{dz}{ds}$$

of $\nabla \mathbf{F}$ and the prefactor **u**.

The scalar and vector of $\nabla \mathbf{F}$ are evidently the divergence and curl of **F**. If the curl is zero the dyadic is symmetric and there is a scalar function ϕ such that

$$\mathbf{F} = \nabla\phi.$$

The dyadic is then

$$\nabla \mathbf{F} = \nabla\nabla\phi$$

from which its symmetry is apparent.

In case of the position vector **R** we have

$$\nabla \mathbf{R} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} = \mathbf{I}.$$

If ∇ is expressed in terms of coördinates u, v, w we then have

$$\nabla u \frac{\partial \mathbf{R}}{\partial u} + \nabla v \frac{\partial \mathbf{R}}{\partial v} + \nabla w \frac{\partial \mathbf{R}}{\partial w} = \mathbf{I},$$

which shows that ∇u, ∇v, ∇w and $\frac{\partial \mathbf{R}}{\partial u}$, $\frac{\partial \mathbf{R}}{\partial v}$, $\frac{\partial \mathbf{R}}{\partial w}$ are reciprocal sets of vectors (cf. §35).

105. Differentiation and Integration of Dyadics. Proceeding as in §12 we get

$$\frac{\partial}{\partial x}\mathbf{AB} = \frac{\partial \mathbf{A}}{\partial x}\mathbf{B} + \mathbf{A}\frac{\partial \mathbf{B}}{\partial x}$$

as the derivative of the dyad **AB**. Except that the order of factors must be preserved the derivative is obtained as in calculus. The derivative of a dyadic is the sum of the derivatives of its dyads.

The divergence and curl of a dyadic are defined by the equations

$$\nabla \cdot \Phi = \mathbf{i} \cdot \frac{\partial \Phi}{\partial x} + \mathbf{j} \cdot \frac{\partial \Phi}{\partial y} + \mathbf{k} \cdot \frac{\partial \Phi}{\partial z}, \tag{453}$$

$$\nabla \times \Phi = \mathbf{i} \times \frac{\partial \Phi}{\partial x} + \mathbf{j} \times \frac{\partial \Phi}{\partial y} + \mathbf{k} \times \frac{\partial \Phi}{\partial z}. \tag{454}$$

If Φ is written in the form

$$\Phi = \mathbf{F}_1\mathbf{i} + \mathbf{F}_2\mathbf{j} + \mathbf{F}_3\mathbf{k},$$

these have the values

$$\nabla \cdot \Phi = (\nabla \cdot \mathbf{F}_1)\mathbf{i} + (\nabla \cdot \mathbf{F}_2)\mathbf{j} + (\nabla \cdot \mathbf{F}_3)\mathbf{k},$$
$$\nabla \times \Phi = (\nabla \times \mathbf{F}_1)\mathbf{i} + (\nabla \times \mathbf{F}_2)\mathbf{j} + (\nabla \times \mathbf{F}_3)\mathbf{k},$$

For $\nabla \cdot \Phi$ to be zero it is necessary and sufficient that

$$\nabla \cdot \mathbf{F}_1 = \nabla \cdot \mathbf{F}_2 = \nabla \cdot \mathbf{F}_3 = 0.$$

The vectors $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ are then solenoidal and so there are three vector potentials $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ such that

$$\mathbf{F}_1 = \nabla \times \mathbf{A}_1, \quad \mathbf{F}_2 = \nabla \times \mathbf{A}_2, \quad \mathbf{F}_3 = \nabla \times \mathbf{A}_3.$$

Consequently

$$\Phi = \nabla \times \Psi$$

where

$$\Psi = \mathbf{A}_1\mathbf{i} + \mathbf{A}_2\mathbf{j} + \mathbf{A}_3\mathbf{k}.$$

In a similar way we show that if

$$\nabla \times \Phi = 0$$

there is a vector $\mathbf{F}$ such that

$$\Phi = \nabla\mathbf{F}.$$

The preceding discussions show that an identity in which a dyadic Φ appears as the last factor in each term is really a set of three identities obtained by replacing Φ by $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ respectively. It follows that *the dyadic relation holds if the corresponding vector equation is an identity.* Thus if ϕ is a scalar function and Φ a dyadic,

$$\begin{aligned}
\nabla \cdot (\phi\Phi) &= \nabla\phi \cdot \Phi + \phi\nabla \cdot \Phi, \\
\nabla \times (\phi\Phi) &= \nabla\phi \times \Phi + \phi\nabla \times \Phi, \\
\nabla \cdot \nabla \times \Phi &= 0, \\
\nabla \times (\nabla \times \Phi) &= \nabla\nabla \cdot \Phi - \nabla^2\Phi.
\end{aligned}$$

For the same reason, if the vector $\mathbf{F}$ and the dyadic Φ are continuously differentiable on a surface S of boundary C,

$$\int_C d\mathbf{R}\,\mathbf{F} = \int_S \mathbf{n} \times \nabla\mathbf{F}\, da, \tag{455}$$

$$\int_C d\mathbf{R} \cdot \Phi = \int_S \mathbf{n} \cdot \nabla \times \Phi\, da, \tag{456}$$

$$\int_C d\mathbf{R} \times \Phi = \int_S (\mathbf{n} \times \nabla) \times \Phi\, da. \tag{457}$$

Likewise if $\mathbf{F}$ and Φ are continuously differentiable in a region V of boundary S,

$$\int_S \mathbf{n}\mathbf{F}\, da = \int_V \nabla\mathbf{F}\, dv, \tag{458}$$

$$\int_S \mathbf{n} \cdot \Phi\, da = \int_V \nabla \cdot \Phi\, dv, \tag{459}$$

$$\int_S \mathbf{n} \times \Phi\, da = \int_V \nabla \times \Phi\, dv. \tag{460}$$

If

$$\oint d\mathbf{R} \cdot \Phi = 0$$

for every closed path in a given region, as in §24 we conclude that the integral

$$\mathbf{F} = \int_{P_0}^{P} d\mathbf{R} \cdot \Phi$$

is independent of the path from P_0 to P and

$$\Phi = \nabla \mathbf{F}.$$

106. Double Multiplication. A dyadic Φ determines a linear transformation of vectors. To each vector $\mathbf{R}$ corresponds a vector

$$\mathbf{R}' = \mathbf{R} \cdot \Phi.$$

To the vectors $\mathbf{R}_1$, $\mathbf{R}_2$ correspond the vectors

$$\mathbf{R}_1' = \mathbf{R}_1 \cdot \Phi, \quad \mathbf{R}_2' = \mathbf{R}_2 \cdot \Phi.$$

There is a dyadic Ψ such that

$$\mathbf{R}_1' \times \mathbf{R}_2' = \mathbf{R}_1 \times \mathbf{R}_2 \cdot \Psi.$$

To express this dyadic in terms of Φ we shall need a new type of product.

The dyad

$$(\mathbf{A} \times \mathbf{C})(\mathbf{B} \times \mathbf{D})$$

is called the **double cross product** of the dyads **AB** and **CD**. Similarly

$$(\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D})$$

is called the **double dot product** of **AB** and **CD**. The double cross product

$$\Phi \overset{\times}{\times} \Psi$$

and the double dot product

$$\Phi : \Psi$$

of two dyadics Φ and Ψ are obtained by multiplying their dyads distributively. That these products are independent of the form in which the dyadics are expressed follows from the remarks in §99 concerning dyadic identities. If a dyadic Φ can be converted into Φ_1 by substitutions of the types (438), (439), (440), the same substitutions will convert $\Phi \times\!\times \Psi$ into $\Phi_1 \times\!\times \Psi$ and $\Phi : \Psi$ into $\Phi_1 : \Psi$.

Suppose

$$\Phi = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \mathbf{A}_3\mathbf{B}_3,$$
$$\mathbf{R}_1' = \mathbf{R}_1 \cdot \Phi = (\mathbf{R}_1 \cdot \mathbf{A}_1)\mathbf{B}_1 + (\mathbf{R}_1 \cdot \mathbf{A}_2)\mathbf{B}_2 + (\mathbf{R}_1 \cdot \mathbf{A}_3)\mathbf{B}_3,$$
$$\mathbf{R}_2' = \mathbf{R}_2 \cdot \Phi = (\mathbf{R}_2 \cdot \mathbf{A}_1)\mathbf{B}_1 + (\mathbf{R}_2 \cdot \mathbf{A}_2)\mathbf{B}_2 + (\mathbf{R}_2 \cdot \mathbf{A}_3)\mathbf{B}_3.$$

In the product $\mathbf{R}_1' \times \mathbf{R}_2'$ the terms containing $\mathbf{B}_1 \times \mathbf{B}_2$ are (cf. equation (27))

$$[(\mathbf{R}_1 \cdot \mathbf{A}_1)(\mathbf{R}_2 \cdot \mathbf{A}_2) - (\mathbf{R}_1 \cdot \mathbf{A}_2)(\mathbf{R}_2 \cdot \mathbf{A}_1)]\mathbf{B}_1 \times \mathbf{B}_2$$
$$= (\mathbf{R}_1 \times \mathbf{R}_2 \cdot \mathbf{A}_1 \times \mathbf{A}_2)\, \mathbf{B}_1 \times \mathbf{B}_2.$$

The entire product is

$$\mathbf{R}_1 \times \mathbf{R}_2 \cdot [\mathbf{A}_1 \times \mathbf{A}_2\, \mathbf{B}_1 \times \mathbf{B}_2 + \mathbf{A}_1 \times \mathbf{A}_3\, \mathbf{B}_1 \times \mathbf{B}_3 + \mathbf{A}_2 \times \mathbf{A}_3\, \mathbf{B}_2 \times \mathbf{B}_3],$$

which can be written

$$\mathbf{R}_1' \times \mathbf{R}_2' = \mathbf{R}_1 \times \mathbf{R}_2 \cdot (\tfrac{1}{2}\Phi \times\!\times \Phi). \tag{461}$$

The reason for the factor $\frac{1}{2}$ is that in the double product $\Phi \times\!\times \Phi$ each term, such as $\mathbf{A}_1 \times \mathbf{A}_2\, \mathbf{B}_1 \times \mathbf{B}_2$, occurs twice.

If then Φ transforms $\mathbf{R}_1$ and $\mathbf{R}_2$ into $\mathbf{R}_1'$ and $\mathbf{R}_2'$, the dyadic

$$\tfrac{1}{2}\Phi \times\!\times \Phi$$

transforms $\mathbf{R}_1 \times \mathbf{R}_2$ into $\mathbf{R}_1' \times \mathbf{R}_2'$.

In a similar way we show that if

$$\mathbf{R}_1' = \mathbf{R}_1 \cdot \Phi,$$
$$\mathbf{R}_2' = \mathbf{R}_2 \cdot \Phi,$$
$$\mathbf{R}_3' = \mathbf{R}_3 \cdot \Phi,$$

the triple scalar products satisfy the equation

$$(\mathbf{R}_1'\mathbf{R}_2'\mathbf{R}_3') = (\mathbf{R}_1\mathbf{R}_2\mathbf{R}_3)(\tfrac{1}{6}\Phi \times\!\times \Phi : \Phi). \tag{462}$$

107. Stress. Consider a region filled with a material which may be solid, liquid, or gas. The different parts of this medium exert forces on each other. Across an elementary area da there is such a force between material on one side and that on the other. This force is evidently proportional to da. Assign a positive direction to the normal and let $\mathbf{T}$ be the force per unit area exerted across da by the medium on the positive side upon that on the negative side. This quantity $\mathbf{T}$ is called the **stress** across the element of area. In general it depends on the position and orientation of the area, and changes sign when the direction of the normal is reversed.

In addition to this stress across a surface in the medium there may be a body force, such as that of gravity, exerted by outside agencies directly on the medium. The magnitude of this force on an elementary volume dv is evidently proportional to dv. Let $\mathbf{F}$ be the value of this force per unit volume. This vector is a function of position and time.

Let ρ be the density of the medium and $\mathbf{A}$ its acceleration in case it is moving. We may apply Newton's law

$$\Sigma(\text{mass})\ (\text{acceleration}) = \text{total force}$$

to the material in any region V. The force exerted across the boundary S of such a region is

$$\int_S \mathbf{T}_n\, da$$

where $\mathbf{T}_n$ is the stress across the element da of the boundary considered as having a normal drawn outward. The force exerted directly on the material by outside agencies is

$$\int_V \mathbf{F}\, dv.$$

Newton's law is thus expressed by the equation

$$\int_V \rho\mathbf{A}\, dv = \int_S \mathbf{T}_n\, da + \int_V \mathbf{F}\, dv.$$

Hence

$$\int_S \mathbf{T}_n\, da = \int_V (\rho\mathbf{A} - \mathbf{F})\, dv, \tag{463}$$

the volume integral being extended through any region and the surface integral taken over its boundary.

Through any point $P(x, y, z)$ pass planes parallel to the coördinate planes and let $\mathbf{T}_x$, $\mathbf{T}_y$, $\mathbf{T}_z$ be the stresses across these planes. By $\mathbf{T}_x$ we mean the force per unit area at P across a plane perpendicular to the x-axis exerted by material of abscissa greater than x on that of abscissa less than x. Similarly let $\mathbf{T}_n$ be the stress at P across a plane with positive unit normal $\mathbf{n}$. We wish to determine the relation between $\mathbf{T}_n$, $\mathbf{T}_x$, $\mathbf{T}_y$, $\mathbf{T}_z$.

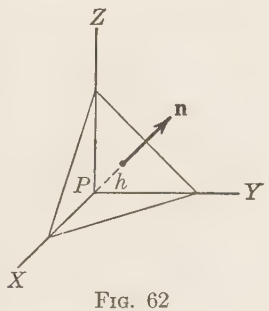


FIG. 62

To do this consider the small tetrahedron formed by planes through P parallel to the coördinate planes and a fourth plane with unit normal $\mathbf{n}$ and perpendicular distance h from P. Let a be the area of the triangle in this fourth plane. Then $(\mathbf{i}\cdot\mathbf{n})a$, $(\mathbf{j}\cdot\mathbf{n})a$, $(\mathbf{k}\cdot\mathbf{n})a$ are the areas of the other three triangles. Applied to this tetrahedron (463) then becomes

$$\overline{\mathbf{T}}_n a - \overline{\mathbf{T}}_x(\mathbf{i}\cdot\mathbf{n})a - \overline{\mathbf{T}}_y(\mathbf{j}\cdot\mathbf{n})a - \overline{\mathbf{T}}_z(\mathbf{k}\cdot\mathbf{n})a = \overline{(\rho\mathbf{A} - \mathbf{F})}\tfrac{1}{3}ha,$$

where the bar over each vector signifies the average value of the vector over the area or volume to which it is applied. Cancelling a we have

$$\overline{\mathbf{T}}_n = \overline{\mathbf{T}}_x(\mathbf{i}\cdot\mathbf{n}) + \overline{\mathbf{T}}_y(\mathbf{j}\cdot\mathbf{n}) + \overline{\mathbf{T}}_z(\mathbf{k}\cdot\mathbf{n}) + \overline{(\rho\mathbf{A} - \mathbf{F})}\tfrac{1}{3}h.$$

Now keep $\mathbf{n}$ unchanged while h approaches zero. In the limit we obtain

$$\mathbf{T}_n = \mathbf{T}_x(\mathbf{i}\cdot\mathbf{n}) + \mathbf{T}_y(\mathbf{j}\cdot\mathbf{n}) + \mathbf{T}_z(\mathbf{k}\cdot\mathbf{n}). \tag{464}$$

Let Φ be the dyadic

$$\Phi = \mathbf{i}\mathbf{T}_x + \mathbf{j}\mathbf{T}_y + \mathbf{k}\mathbf{T}_z. \tag{465}$$

Equation (464) can then be written

$$\mathbf{T}_n = \mathbf{n}\cdot\Phi. \tag{466}$$

The quantity Φ is called the **stress dyadic** and (466) shows how the stress across a plane of positive unit normal $\mathbf{n}$ is related to it.

By integration over a closed surface we get

$$\int_S \mathbf{T}_n \, da = \int_S \mathbf{n} \cdot \Phi \, da = \int_V \nabla \cdot \Phi \, dv,$$

the volume integral being extended through the interior. The quantity

$$\nabla \cdot \Phi = \frac{\partial \mathbf{T}_x}{\partial x} + \frac{\partial \mathbf{T}_y}{\partial y} + \frac{\partial \mathbf{T}_z}{\partial z} \tag{467}$$

is the divergence of the stress dyadic. Equation (463) is equivalent to

$$\int_V \nabla \cdot \Phi \, dv = \int_V (\rho \mathbf{A} - \mathbf{F}) \, dv.$$

Since this is valid for every region, the elements of integration on the two sides are equal and so

$$\rho \mathbf{A} = \nabla \cdot \Phi + \mathbf{F}. \tag{468}$$

When $\mathbf{F}$ and Φ are given this equation determines the acceleration of the medium.

The dyadic Φ cannot however be assigned arbitrarily. For the rate of change of angular momentum about any point is equal to the moment of external forces about that point. If V is a region of boundary S and $\mathbf{R}$ is the vector from any point P to the element of integration we then have

$$\int_V \rho \mathbf{R} \times \mathbf{A} \, dv = \int_S \mathbf{R} \times \mathbf{T}_n \, da + \int_V \mathbf{R} \times \mathbf{F} \, dv.$$

By means of equations (466) and (468) this can be reduced to the form

$$\begin{aligned} \int_V \mathbf{R} \times \nabla \cdot \Phi \, dv &= \int_S \mathbf{R} \times (\mathbf{n} \cdot \Phi) \, da \\ &= -\int_S \mathbf{n} \cdot \Phi \times \mathbf{R} \, da = -\int_V \nabla \cdot (\Phi \times \mathbf{R}) \, dv. \end{aligned}$$

Since this holds for every region V

$$\mathbf{R} \times \nabla \cdot \Phi = -\nabla \cdot (\Phi \times \mathbf{R}) = \mathbf{R} \times \nabla \cdot \Phi + \Phi_v.$$

Consequently

$$\Phi_v = \mathbf{i} \times \mathbf{T}_x + \mathbf{j} \times \mathbf{T}_y + \mathbf{k} \times \mathbf{T}_z = 0. \tag{469}$$

Equation (469) expresses that the stress dyadic is symmetric (§101). At each point P of the medium a set of principal axes can then be chosen such that

$$\Phi = a\mathbf{ii} + b\mathbf{jj} + c\mathbf{kk}. \tag{470}$$

Across a plane perpendicular to $\mathbf{i}$ the stress is

$$\mathbf{i} \cdot \Phi = a\mathbf{i}.$$

If a is positive this is a tension parallel to $\mathbf{i}$. If a is negative it is a compression. In the neighborhood of a point the stress is thus equivalent to three tensions or compressions in perpendicular directions. At different points these directions need not be parallel.

INDEX

The numbers refer to the pages

A

Acceleration, 20
Addition of vectors, 2
Analytic function, 155, 207
Analytic properties of harmonic functions, 153
Angular momentum, 22
  velocity, 12
Antecedents of a dyadic, 216
Anti-symmetric dyadic, 217
Arc, regular, 49
Associative law, of addition, 3
  of multiplication, 222
Average value theorem, 156
  converse of, 174

B

Bounded function, 32

C

Center of gravity, 23
Charges, point, 118
  space, 120
  surface, 135
Circulation, 55
Closed region, 32
Commutative law, of addition, 2
  of multiplication, 7
Component, normal, 106
  of a vector, 6, 7
  tangential, 106
Conjugate dyadics, 217
Consequents of a dyadic, 216
Conservative force, 54
Continuity, 31
  equation of, 192, 212
  uniform, 31
Continuous function, 31
  piecewise, 32
Continuously differentiable, 32
Convergence of improper integrals, 123
Coördinate curves, 80, 91
  surfaces, 80
Coördinates, curvilinear, 80
  cylindrical, 85
  on a surface, 91
  orthogonal, 84
  rectangular, 6
  spherical, 86
Coulomb's law, 118
Couple, 14
Curl, of a vector, 43, 90
  of a dyadic, 224
Current, space, 192
  surface, 192
Curve, regular, 49
Curvilinear coördinates, 80
Cylindrical coördinates, 85

D

Del (∇), 35
  in general coördinates, 88
  ∇_P and ∇_Q, 116
  ∇_t, 201
Derivative, directional, 33
  normal, 151
  of a dyadic, 224
  of a vector, 19, 223
Determinants, 12, 16, 41
Differentiable, continuously, 32
  piecewise, 32
Differential equations, 20
Differentiation formulas, 38, 225
Directional derivative, 33
Dirichlet problem, 165
Discontinuities, 106, 179
Distributive law of multiplication, 4, 8, 12, 222

Divergence, of a dyadic, 224
of a vector, 43, 89
theorem, 67, 96
Double layer, 141
potential of, 141
vector potential of, 142
Double products, 226
Doublet, 140
Dyad, 216
Dyadic, 216
anti-symmetric, 217
conjugate of, 217
curl of, 224
differentiation and integration of, 224
divergence of, 224
reciprocal of, 222
scalar of, 219
stress, 229
symmetric, 217
vector of, 219
Dyadics, products of, 221, 226
Dynamics of a particle, 21

E

Electric intensity, 111
due to a double layer, 142
due to a point charge, 118
due to a space charge, 121, 127
due to a surface charge, 135, 138
in two dimensions, 144
Electromagnetic field, 213
Electrostatic field, 111
Element, of arc, 85
of volume, 84
Euler's theorem on homogeneous functions, 172

F

Faraday's law, 213
Field, 30
electromagnetic, 213
electrostatic, 111
two-dimensional, 144
variable, 200
Flow across a surface, 66
Force, conservative, 54
moment of, 13, 14
work done by, 9
Formulas, differentiation, 38, 225
integration, 72, 225
Functional relations, 40
Functions, bounded, 32
continuous, 31
continuously differentiable, 32
harmonic, 149
linear, 4
linear vector, 215
of one scalar variable, 18
of position, 30
piecewise continuous, 32
piecewise differentiable, 32
uniform, 30
uniformly continuous, 31

G

Gradient, 32, 88
surface, 91
Green's function, 162
for the sphere, 167
of the second kind, 171, 172
properties of, 165

H

Harmonic, at a point, 149
in a region, 149
outside a surface, 157
Harmonic function, 149
analytic properties, 153
average value properties, 156, 174
boundary value properties, 150, 161
in an infinite region, 157
maximum and minimum properties, 160
representation as potential, 176
Helmholtz's theorem, 187
Hölder condition, 132, 137
Huygens' principle, 206

I

Idemfactor, 220
Indeterminate product, 217
Infinite region, 157, 187

Integrability, condition of, 102
Integrals, improper, 122
  line, 50
  surface, 61
  volume, 90
Integration formulas, 72, 225
Interior point, 32
Irrotational vectors, 100, 195

J

Jacobian, 42

L

Laplace's equation, 132
  in cylindrical coördinates, 150
  in orthogonal coördinates, 150
  in spherical coördinates, 180
Limit, 18
  at infinity, 31, 157
Line integral, 50
  independent of path, 51
Linear function, 4
Linear vector differential equations, 20
Linear vector function, 215
Logarithmic potential, 145

M

Middle factor, rule of, 17
Moment, of a couple, 14
  of a doublet, 140
  of a force, 13, 14
Momentum, 21
  angular, 22
Multiplication, of dyadics, 221, 226
  of scalar and vector, 3
  of two vectors, 7, 10, 217

N

Neumann's problem, 170
Normal component, 106
  derivative, 151
  to a surface, 61

O

One-valued function, 30
Open region, 32
Orthogonal coördinates, 84

P

Particle, dynamics of, 21
Piecewise continuous, 32
  differentiable, 32
Poisson's equation, 131
  integral, 169
Postfactor, 216
Potential, logarithmic, 145
  of a double layer, 141
  of a point charge, 120
  of a space charge, 121, 127
  of a surface charge, 135, 137
  retarded, 203
  scalar, 54, 182, 188
  two-dimensional, 145
  vector, 104, 188
  velocity, 101
Potentials of a vector, 188, 190
Prefactor, 216
Product, double, 226
  indeterminate, 217
  of dyadics, 221
  of scalar and vector, 3
  of unit vectors, 12
  scalar, 7
  triple, 15, 16
  vector, 10

R

Reciprocal of a dyadic, 222
Reciprocal sets of vectors, 82, 220
Rectangular coördinates, 6
Reducible curves and surfaces, 100
Region, closed, 32
  infinite, 157, 187
  open, 32
  simply connected, 100
Regular arc, 49
  at infinity, 159, 187
  curve, 49
  point on a surface, 59
  surface element, 59

Retarded potentials, 203, 210
in terms of characteristics, 210
Rotation, about an axis, 12
right-handed, 6

S

Scalar, 1
multiplication by, 3
of a dyadic, 219
potential, 54, 182, 188, 201
product, 7
Simply connected region, 100
Solenoidal vectors, 103, 195
Solid angle, 112
derivatives of, 117
discontinuities, 117
subtended by a closed surface, 114
total about a point, 112
Sources and sinks, 70
Space charge, 120
current, 192
Spherical coördinates, 86
Stokes's theorem, 62
Stress, 228
dyadic, 229
Subtraction of vectors, 3
Surface charge, 135
curl, 93
current, 192
discontinuities, 106
divergence, 93
divergence theorem, 96
element, regular, 59
gradient, 91
integral, 61, 94
positive direction along border of, 55
Symmetric dyadic, 217
principal axes of, 219

T

Tangential component, 106
Torque, 22
Triple products, 15, 16

U

Uniform continuity, 31
function, 30
Unit vectors, i, j, k, 6
products of, 12

V

Vector, 1
as sum of irrotational and solenoidal vectors, 186
characteristics which determine, 177
curl of, 43
derivative of, 19, 223
discontinuities of, 106, 179
divergence of, 43
in terms of potentials, 188, 209
irrotational, 100, 195
of a dyadic, 219
potential, 104, 188
product, 10
solenoidal, 103, 195
Vectors, addition of, 2
equality of, 1
multiplication of, 7, 10, 217
reciprocal sets of, 82
subtraction of, 3
Velocity, 19
angular, 12
potential, 101
Volume integral, 90
Vortex, 65

W

Wave equation, 206
function, 206
Work, 9